

Is There any Impact of Teaching Vector Spaces From Real Problems? The Case of First Year Engineering Students

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Abstract: In some linear algebra courses at the university level in engineering majors, the vector spaces are presented to students in an abstract way with scarce connections with other subjects and real problems. The goal of this study was to examine the effectiveness, regarding content knowledge and motivation, of a didactic proposal based on a problem based learning and the necessity principle, PBL-NP, modelling real engineering problems through homogeneous systems of linear equations, to introduce the concept of vector space. A quasi-experiment (post-test) was designed with a convenience sample composed of two groups: the experimental group, EG, amounting 33 students who were taught using the PBL-NP, and the control group, CG, composed by 79 students, taught by following an abstract approach. Inferential statistics was used to compare the learning outcomes between groups, by using as contrast variable an external test. The results show that the students in the EG group felt more relaxed and put less effort than CG students, while both groups gather the abstract concepts in a similar extent. Also the percentage who passed the course is higher in the EG compared with CG. Although both groups value positively the subject, a percentage of students in the CG group add some comments referred to the lack of practice related with real problems in the algebra courses taught with the abstract approach.

Keywords: *achievement; linear algebra; motivation; necessity principle; problem solving; vector spaces*

Introduction

At university level in engineering schools, algebra courses are a challenge from years and entails some research (Dorier, & Sierpiska, 2001; Dorier et al., 2000; Wang, 1989). There is a way to teach linear algebra at universities for applied sciences and engineering, based on introducing the formal theory of vector spaces by following a whole conceptual design (Burgos, 2000; Groszman, 1995). Definitions, theorems, properties, are introduced firstly, and the examples and mechanical exercises are introduced secondly, when time allows that. We could name this

way of teaching abstract methodology (AM). From this teaching perspective the students are asked to learn abstract formalisms without any context, disconnected from other subjects and from real world problems, and presented in a forced way, without generating inside their minds any intellectual need (Harel, 2000). The teacher's task in teaching linear algebra is arduous as well because he or she perceives this lack of meaning for students, what usually entails a lack of interest or motivation in students (Dorier, 1998; 2000). This abstract way of teaching could be considered close to the one named in other papers as

traditional (Clark & Dickerson, 2018; Mason et al., 2013)

The present proposal consists of teaching the concepts of vector spaces and subspace starting from real life problems, usually afforded by other sciences, and use the question as a tool. The questions are posed in such a way that require the students to discover the theoretical concepts when answering the practical questions. Such an approach, we hope, motivate the students, stimulate them and serves as a guide with the material, so that they can build meaningful knowledge on their own. Also, the teacher feels happier with the course. Therefore, the main objective of this paper will be to compare between the proposed and the abstract way of teaching vector spaces.

The teaching methodology evaluated in this paper involves problem-based learning, PBL (Mills & Treagust, 2003), and the necessity principle. How does PBL contribute to deep knowledge and the development of metacognition? Learning by PBL has a profound effect on performance, and on understanding and comprehension (Collins & Ferguson, 1993; Day, 1988; Zhang, 1997). PBL involves not only the use of concepts or knowledge, but also the development of executive functions and self-regulation, of vital importance for the future professional life of engineering students. In addition, it contributes to the development of mental flexibility, a crucial aspect of adaptation and intelligence in a wide sense, also connected to creativity (Smith, 1983).

Theoretical framework

The concept of vector is assumed to be *embodied* by Watson et al. (2003). These authors referred to the three levels between which the concept apprehension

transits: embodied, proceptual and formal. The first one linked with senses perception and physical interactions. The second, arises when introspection and idea generation vanishes the direct connection with real world, sometimes by using representations or visualization (Konyalioglu et al., 2003; Konyaloğlu et al., 2005); and the third is exactly mathematical symbolism. This applies to the vector concept which has a different treatment in physics or mathematics. On the one hand, vectors are embodied in physics, representing magnitudes with direction such as velocity, force, acceleration, that are added nose to tail; becomes a *procept* with the idea of translation, involving a representation of that vector through a column or row matrix. Finally, the symbolism the formal concept of vector is defined in terms of a vector space which consists of a set of elements with certain properties where the addition is defined as an operation. On the other hand, in mathematics a vector inhabits the embodied world and the proceptual world at preuniversity level, where it is represents a translation of any geometrical object in a plane, represented by just a segment with direction. Nonetheless, at university level there is a big jump when starting algebra courses where vectors are used as vector spaces and subspaces, presented and afforded by formal instruction (Konyalioglu et al., 2003), without strengthening the connection with the embodied and proceptual worlds. It is the student who must link the second with the third level, in most cases by his/her own, because the instruction at university level in most cases is not fostering or facilitating the creation of this connection. The vector concept is assumed to be subtly in the intersection of the embodied theory relating to physical phenomena, and process-object encapsulation of actions as mathematical concepts (Watson et al., 2003).

Therefore, efforts are needed to improve the leaning of this concept, as well as the vector space concept as a generalization of the vector concept.

In algebra courses in engineering schools it's being a challenge to teach effectively formal concepts. Several authors research on the most effective way to teach algebra in engineering schools. Among them, Mills and Treagust (2003) proposed discard to use mainly mathematical symbolism, and realized as the most appropriate the problem-based learning (PBL) or project-based learning, assuming that both kinds of instruction do exhibit strengths and weaknesses. The vector space concept taught through PBL, and its confrontation with the abstract methodology (or algorithmic) one has been widely studied in the field of freshman university chemistry (Nakhleh & Mitchell, 1993; Nurrenbern & Pickering, 1987; Sawrey, 1990). This contrast has also been studied in the case of university algebra courses regarding the assessment of cognitive and metacognitive strategies (Bayat & Tarmizi, 2010; Tarmizi & Bayat, 2010; Julian, 2017). In all these studies, the advantage was for the PBL teaching compared with the abstract way of teaching mathematics in general, or algebra, specifically. It could be since chemistry, like the other sciences, do use mathematics and algebra at any specific situation where the concepts should be applied. That is also the case of engineering majors, where algebra is a tool to solve real world problems as a final purpose. Therefore, to teach theoretical concepts isolated from practice does not seem a way to promote global engineer skills in engineering students.

The other key aspect of the proposed methodology is the necessity principle. According to Harel (2000) the

Necessity principle states that "...for students to learn, they must see an intellectual need (as opposed to social or economic) for what they are intended to be taught" (p 177). It is based on the Piagetian assumption that knowledge is developed as a solution to a problem. In this sense troubles are produced to question the results to introduce the main concepts.

Another aspect that can influence learning, of interest for authors in this paper, is student motivation. The motivation has widely been analyzed referred to mathematics, but just a few studies are devoted to confronting alternative methodologies with the abstract one in teaching algebra, by means of achievement and motivation (Jing et al., 2017; Ting et al., 2018; Toussaint, 2016). Among them, only the last one is focused on university students, but not of engineering majors, who have not been studied in this respect. The possible effect of different methodologies on motivation is controversial: while Toussaint (2016) concluded that specifically academic service learning has a positive effect on both, the achievement and the students' motivation, in the other references (Jing et al., 2017; Ting et al., 2018) authors do not find any effect for the Variation Theory, except in the achievement.

Research question

The goal of this study was to examine the impact of presenting the algebraic concepts of vector space and subspace starting from real world problems related with engineering studies, under a problem based perspective provoked from the necessity principle (PBL-NP). The research question is: what is the impact of this methodology in the teaching-learning of these concepts in the students? That is, how students perceive value/usefulness, interest/enjoyment,

effort/importance and pressure/tension, and does it influence their achievement?

Methods

The method is a quasi-experiment with control and experimental groups, non-probabilistic sampling, and post measurement of the dependent variables: motivation and achievement (Campbell & Stanley, 2015).

Participants and context

Three different course-groups of first year Engineering students of Castilla La Mancha University were compared. The algebra course met four times a week during the semester in one-hour sessions. One of the groups had 33 students whose professor was also a researcher (Julian, 2017), taught through the PBL-NP approach, constituting the experimental group (EG). The other two course-groups were composed by 40 and 39 students and were taught by other professors. They used mainly an abstract approach, AM, mentioned in the introduction, implying the usual presentation of definitions, properties, theorems, and at the end, and just a few, examples, and mechanical exercises, without organizing tutorial groups. There was no difference between the groups. As a whole, these two course-groups served as the control group (CG).

Materials

The materials for the vector spaces and subspaces class with the EG group were developed under a PBL and the necessity principle perspective emphasizing the practical implications of base and linear combinations, and consisting on:

- Engineering examples of problems involving homogeneous systems of algebraic equations.
- Class development and connection with vector spaces.
- Exercises to explain and consolidate the concept. The students worked in class with the teacher support, and there, the exercises and problems were corrected.

One of the designed proposals is included as example in Appendix 1.

In order to test the efficiency of the proposal, a post-test was delivered to the students in both groups. The post-test is a form composed by questions and several sub questions. The first question contains 12 items to measure motivation, extracted from the Intrinsic Motivation Inventory (IMI), of public access on the internet at <https://selfdeterminationtheory.org/intrinsic-motivation-inventory/>. The psychometric properties of this instrument have been reported since 1989 by McAuley, Duncan and Tammen, including the confirmatory factor analysis. In the instrument, items 1.1-1.7 measure value/usefulness; items 1.8-1.9 measure interest/enjoyment; items 1.10-1.11 measure effort/importance; and item 1.12, measures pressure/tension. The answers are posed by following a 5-points Likert scale, from totally disagree (1) to totally agree (5) crossing neutrality (3). In addition, several questions have been included in the form, in such a way that Question 2 measures the perception on the usefulness of learning some concepts of algebra; Question 3 consists of two theoretical questions on the subject, as serves to evaluate content knowledge; and Question 4 is a free question asking for comments about the teaching of the subject. The specific form with items and questions appears in Appendix 2. To

receive a course grade, a written exam was compulsorily offered to students, the question on vector spaces included in for each group is shown in Appendix 3. The score in this exam is used as a measure of achievement.

Procedure

The experimental group was taught according with the proposal starting from engineering problems modelled with homogeneous systems of equations. The control group was taught with the abstract approach. A motivation and content knowledge post-test were delivered to both groups several weeks after the subject of vector spaces was studied in the classes. At the end of the course, both groups do perform the final exam, including the question shown in Appendix 3.

Statistical analysis

The possible differences between the two groups in the answers to the questions, and the exam scores were analyzed. The data obtained from the test were compared by using the following statistical techniques: Pearson Chi-square normality tests, comparison of means and proportions. The level of significance considered was 0.05. The data do not follow a normal distribution, as confirmed by the Pearson Chi-square normality tests, since the p -values are lower than 10^{-7} for the EG group and p -values are less than 10^{-16} for the CG group. Therefore, a nonparametric Kruskal-Wallis (K-W) test is performed. On the other hand, because the EG and CG are independent groups with sufficiently large sample size, a non-parametric test of percentage difference is performed.

Results

In table 1 are included the values of the mean for each item in the motivation post-test with the corresponding p -value for EG and CG. Only the p -values for items 1.10 and 1.12 are lower than .05, therefore there are significant differences between EG and CG students only for these items.

Table 1

Mean for Each Item and p-value for the Mean Test in EG and CG.

| Item/question | EG | CG | p-value |
|---------------|------|------|---------|
| 1.1. | 3.64 | 3.84 | .2552 |
| 1.4. | 3.34 | 3.11 | .4078 |
| 1.6. | 3.63 | 3.81 | .2138 |
| 1.7. | 3.94 | 3.89 | .8674 |
| 1.8. | 3.44 | 3.66 | .0941 |
| 1.9. | 3.55 | 3.48 | .6047 |
| 1.10. | 3.28 | 3.75 | .0042 |
| 1.11. | 4.44 | 4.37 | .9633 |
| 1.12. | 3.75 | 3.28 | .0281 |

Although the agreement with the items is required as a Likert scale from 1 to 5, there are some free answers emitted by students. The different answers have been categorized by considering the semantic meaning. The translation from Spanish of the students' suggestion or proposals generated these five categories: *problem solving*, *knowledge*, *other subjects*, *future*, and *do not know/do not answer*. Then, relative frequency as the percentage is considered for each category. In table 2, the categories and the percentage for each are shown. All the p -values comparing percentages are larger than 0.05, therefore there are not significant differences for EG and CG students. Despite that, for the *future* ($p=0.056$) category the p -value is really close to the threshold, which can be considered marginally significant.

Table 2

Percentages of Free Answers Provided by Students to Items 1.2, 1.3-1.5

| | EG | EG | EG | CG | CG | CG |
|-------------------|-----|-----|-----|-----|-----|-----|
| | 1.2 | 1.3 | 1.5 | 1.2 | 1.3 | 1.5 |
| Problem solving | 9 | 15 | 21 | 11 | 15 | 11 |
| Knowledge | 18 | 45 | 27 | 27 | 25 | 27 |
| Other subjects | 18 | 9 | 18 | 16 | 8 | 13 |
| Future | 24 | 6 | 3 | 37 | 21 | 19 |
| Don't Know/Answer | 18 | 24 | 30 | 13 | 32 | 30 |

For Question 2, which measures the perception on the usefulness of learning some concepts of linear algebra, only the p-values for questions 2-6-2.8, basis of a vector space, dimension of a vector space, and coordinates of a vector in a basis are lower than .05 ($p=.0020$; $p=.0393$; $p=0.0059$, respectively), exhibiting significant differences between EG and CG.

In Question 3, which evaluates the content knowledge by means of two theoretical questions on the subject, vector spaces and subspaces, 21% of the students from EG and 27% from CG answered correctly, without exhibiting significant difference. Regarding question 4, the free question asking for comments about the teaching of the subject, 15% of the students from the EG and 24% from the CG answered, again without any statistical significance. The answers to question 4 are shown in Table 3.

At the end of the course students took the course exam, whose score was considered the achievement. The percentage of students who passed the course in the first call was 90% for the EG and 33% for the CG ($p=10^{-7}$). Therefore, this variable shows difference in both groups, exhibiting better achievement in the EG.

Table 3

Summary of Answers to Question 4.

Experimental group: 5 students responded (15%)

| |
|--|
| 1 it is difficult to understand. spend more time |
| 2 when someone leaves the board go explaining. not at the end |
| 3 explain the concept of space and vector subspace well |
| 4 try to make what we study tangible because I am not able to mechanize something that I do not understand |
| 5 explain with more examples |

Control group: 19 students responded (24%)

| |
|---|
| 1 more examples among the theory |
| 2 difficult for the bad base. teachers should impart with more enthusiasm so that we become more interested |
| 3 more problems |
| 4 more interest in the main definitions such as vector spaces |
| 5 more exercises and step by step |
| 6 more examples to the explanations |
| 7 spend more time on problems and explain with different examples the different issues and difficult concepts |
| 8 more hours of class. it goes very fast for how abstract it is |

9 more problems. the distribution between theory and problems is inefficient

10 more exercises solved

11 explain real applications that have vector spaces and algebra. not just how to calculate them or how to work with them

12 that the explanations should be not so mathematical

13 geometric images of what we do more often. to visualize subspaces. linear transformations or projections. the fourth dimension. areas of figures. distances ... more pragmatic things

14 none. has been done correctly

15 more exercise classes would improve comprehension

16 the issue of Euclidean space is the most complex and it is very difficult to explain and understand 100%. the other issues if they are well understood

17 don't give the classes so theoretical and make many more examples

18 give more practical examples to realize how useful it will be for us in the future

19 more classes of problems

content knowledge, a post-test was delivered to students. The achievement was measured by the score in the final course exam.

The findings for the motivation analysis showed that there are not differences in items 1.1-1.7, since both groups appreciated the value and usefulness of the subject. The students answered the free questions 1.2, 1.3 and 1.5 with statements or suggestions falling in one of the five categories problem solving, knowledge, other subjects, and do not know/do not answer, in percentages without significant differences, while in the future category the difference is marginally significant. It could show that students in the CG didn't perceive the future use of vector spaces in a higher extent than students in the EG, which could be due to the PBL-NP instruction in the last one. Items 1.8-1.9 measured interest/enjoyment, and no differences for those items were found. Items 1.10-1.11 measured effort/importance, and a difference was observed in such a way that the experimental group declared to put less effort than the control group. Item 1.12 considers pressure/tension, showing a difference in this item, with EG students declaring having less pressure than these in the CG. These findings are aligned with results of House and Telese (2008) about the influence on the students' beliefs and their achievement in algebra courses.

Discussion

The goal of this study was to examine the impact of presenting the algebraic concepts of vector space and subspace starting by modelling realistic problems, PBL, related to the necessity principle to engineering students. A quasi-experimental design was used to compare two groups of algebra in the first year of engineering studies. To measure motivation and

Regarding Question 2, which measures the perception on the usefulness of learning some concepts of linear algebra, there are no differences in questions 2.1-2.5, since both groups perceive the importance of learning those concepts. Contrary, there is a difference in the perception of the concepts of basis, dimension, and coordinates, for which CG appreciated more importance than EG. The results for Question 3, who

measured the content knowledge by means of two theoretical questions on the subject, and the percentage of students that answer correctly the technical question related with the subject is equal in both groups, despite the scores in the final course exam (achievement), is higher in the EG.

This fact resulted surprising for authors, who expected to collect evidence on different motivation caused by instruction, as well as on achievement. Nonetheless, in our quasi experiment there are many strange variables that have not been controlled. One of them is the lack of knowledge about the starting point of both groups regarding the concerned variables, motivation and content knowledge. It could be that the students entering engineering schools, provided that the access is hard in Spain, do incorporate an intrinsic motivation towards algebra which is not that affected by instructions issues. On the contrary, although in the topic related questions there is not difference, the scores in the final course exam (achievement) show significant difference, as expected, in line with the findings of other authors (Jing et al., 2017; Ting et al., 2018).

About Question 4, dealing with comments about the teaching of the subject, only 15% in the experimental group answered that question and included some comments on teaching, while 24 % in the control group. Most of the comments in the control group related with the preference for a more practical course, four of them requesting problems applied to real life, and the rest asking for a course not so theoretical. Even, one of them was asking for more enthusiastic teachers and teaching in order the students to be more interested. These answers can be seen in Table 3. There are just a few comments on improvement

suggestions from EG students, while there are much more from students in CG. Remarkable are the comments 2, 11-13, 17-18 in the CG, which evidence the students' perception of the lack of connection with real life problems and with other subjects, the scarce practical examples and connection with geometrical figures or projections that could support students mental modelling, and the poor emphasis on the future use of the topic. It is also remarkable the statements that evidenced the students' perception of the professor motivation because some referred literally "... teachers should impart with more enthusiasm so that we become more interested". It might relay on lecturer related effects (Hattie, 2012; Rojas & Delofou, 2015) although in this case we consider that the learning environment design attracts students and make the teacher drive the class with more enthusiasm.

To summarize, on the one hand, regarding motivation, some items do not show any difference in EG and CG students, since students got similar results in interest and technical questions in both groups. In addition, all the students value the subject, because they said it is useful. The findings are aligned with the results of Jing et al. (2017). It could be since engineering students, specifically in Spain, are motivated to hard work with mathematics and algebra before entering the university. More studies are required to confirm, or not, this hypothesis. On the other hand, we could say that students' self efficacy is more positive in the EG provided that they felt doing less effort and studied less than the ones in the control group. Nonetheless, some of the students only in CG would like to know the applications, which made researchers infer that the students in EG had enough of them. As Harel (2000) recommended, the inclusion of practical situation for the learning of algebra makes it more meaningful for

students. Although there are many research on teaching proposals for first year university algebra courses (Mills & Treagust, 2003; Nakhleh & Mitchell, 1993; Nurrenbern & Pickering, 1987; Sawrey, 1990), unfortunately are still not abundant enough in most engineering schools. Regarding the difference in the students scores in the final course exam, it is statistically significant and bigger in EG. Additionally, the EG group professor reported a warm classroom atmosphere, challenging for students, whose results motivated them to solve the problems. This learning environment design attracts students and gets the teacher with more enthusiasm. From these comments it can be inferred that for the EG professor the affects in the classroom are crucial to carry out their job, while it is not considered in many mathematics courses at university level (Toussaint, 2016). This evidence entails the importance and influence of teacher on the learning atmosphere (Hattie, 2012; Rojas & Delofou, 2015).

Conclusion

To conclude, the proposal to teach vector spaces and subspaces based on the necessity principle and problem based learning introduced here contributes to improve the class atmosphere, and rises the self-efficacy of students because they stated to perform less effort in affording the topic, as well as to feel less pressure in the course. Moreover, the achievement measured as the score in the final exam is higher in this group, resulting in a higher number of students passing the course.

Nonetheless, this study has limitations which make the results not widely generalizable. The main point is that the design could be improved by collecting information about the original situation by means of a pretest for the motivation and achievement. Therefore, it could be considered a pilot study susceptible of provoking a wider analysis with a pre-post test quasi-experiment design, and several of the strange variables controlled. This incorporations would let to check the generalizability and usefulness of the proposal for first year engineering algebra courses.

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Appendix 1

- a. Engineering examples.

Linear equations systems arise naturally when scientists or engineers study some network flows. A network consists of a collection of nodes connected by lines or arcs called branches. The flow direction is indicated in each branch, as well as the flow, both being part of a vector variable. The basic assumption in flow networks is that the flow entering it is the same as the flow leaving the node. The network analysis determines the current flow in each branch when only some partial information is known or provided (inputs to the network). In the network in figure 1 the traffic flow for several one-way streets in Ciudad Real, in vehicles per hour, is shown. The task is to determine the overall flow pattern for the network, write down the equations describing the flow, and then find the general solution for the system. We assigned the street intersections to nodes, and the unknown flows in the branches (the dependent variables) as x_1, x_2, x_3, x_4 and x_5 , as depicted in figure 1. At each intersection, we set the income flow equal to the output flow. At the node between Inmaculada Concepción and Bachiller Fernán-Gómez streets, flow continuity requires to set the equation, $x_1 + x_2 = x_3$. At the Santa Teresa and Inmaculada Concepción node the resulting equation is $x_3 = x_4 + x_5$

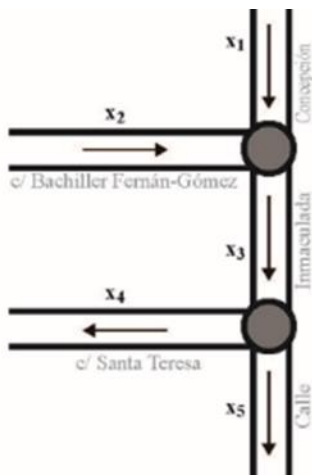


Figure 1. Flow network.

Then, we wrote the system passing all terms to the left

$$x_1 + x_2 - x_3 = 0; x_3 - x_4 - x_5 = 0;$$

and in matrix form as follows

$$\begin{pmatrix} 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Calling B the matrix in this system, the matrix equation would be $B \times X = 0$. Provided that the rank of B is 2, and there are five unknowns, the system will be indeterminate compatible comprising infinite solutions written in terms of three parameters: x_2 as α , x_3 as β and x_4 as γ , where α , β , and γ are real numbers. Once this is done, a general

solution is obtained, which is the following: $x_1 = -x_2 + x_3 = -\alpha + \beta, x_2 = \alpha, x_3 = \beta, x_4 = \gamma, x_5 = x_3 - x_4 = \beta - \gamma$.

If the solution is written in the vector form, equation 4 is obtained.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -\alpha + \beta \\ \alpha \\ \beta \\ \gamma \\ \beta - \gamma \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

In the expression above, the vector $X = (x_1, x_2, x_3, x_4, x_5)$ is a linear combination, of the vectors $u_1 = (-1, 1, 0, 0, 0)$, $u_2 = (1, 0, 1, 0, 1)$, and $u_3 = (0, 0, 0, 1, -1)$, which are the base of the vector space comprising the solutions of this node. As the streets considered in this problem are one-way, none of the variables can be negative, which entails several constraints for the solution values. If, for instance, we give values like $\alpha = 10, \beta = 20$ and $\gamma = 5$, the solution is $v = (10, 10, 20, 5, 15)$.

b. Class development and connection with vector spaces.

We consider the set of solutions of the previous problem:

$$H = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \mid x_1 + x_2 - x_3 = 0, x_3 - x_4 - x_5 = 0\} = \{(x_1, x_2, x_3, x_4, x_5) = \alpha(-1, 1, 0, 0, 0) + \beta(1, 0, 1, 0, 1) + \gamma(0, 0, 0, 1, -1), \alpha, \beta, \gamma \in \mathbb{R}\}$$

And we work with this set, we ask to the students

1. Write three different solutions of the problem. The teacher asks if some vectors belong to this set or not. These are questions to manipulate the set and to know better this set.
2. If we add to the system the global flow equation: $x_1 + x_2 = x_4 + x_5$, we get a system of three equations and five unknowns. Is this system equivalent to the previous one? i.e., both have the same solutions? In the new system the equations are linearly dependent. These are questions to introduce the concept of linear dependency.
3. If I solve the new system I get, for instance: x_2 as α, x_4 as β and x_5 as γ , where α, β , and γ are real numbers. Once this is done, a general solution is obtained, which is the following: $x_1 = -\alpha + \beta + \gamma, x_2 = \alpha, x_3 = \beta + \gamma, x_4 = \beta, x_5 = \gamma$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -\alpha + \beta + \gamma \\ \alpha \\ \beta + \gamma \\ \beta \\ \gamma \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Is this solution correct? Are both solutions correct? This is a linear combination of three vectors.

We could describe the solutions as a linear combination of the vectors in the first question, this solution would be correct? This question and the next one introduces linear dependency of vectors, generated space, basis, dimension and coordinates.

4. Given two solutions, is solution the sum of these solutions?
5. Given a solution, is solution the product of a real number by a solution?
6. What operations are we using in H? This question and the two previous ones introduce the concepts of vector subspace and space.

Students do not know the answers to questions 2 and 3. These questions were used to introduce an intellectual need for the new concepts under the necessity principle. They give good answers to questions 1, 4-6 to construct the concepts of vector space and subspace.

Once the engineering problem has been worked out, and some examples of vector spaces are presented, mainly $\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^5, \dots, \mathbb{R}^n$, the usual way of teaching is followed: definition of vector space and subspace, equations of vector subspaces, properties, theorems, examples and exercises, among them we go back to the engineering problem, and we study the existence of solutions of linear systems as the independent term belongs to the space generated by the

columns of the associated matrix. Next, we define linear dependency, properties, theorems, examples and exercises, among them we go back to the engineering problem. We define basis, dimension and coordinates, properties, theorems, examples and exercises, among them we go back to the engineering problem.

c. Example of exercises.

(1) Determines whether each of the following subsets of \mathbb{R}^3 is a vector subspace:

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}; \quad T = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0 \text{ or } y = z\}$$

(2) Determines the parametric and cartesian equations of the following subspaces:

$$W = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}; \quad Y = \{(t, 2t) \in \mathbb{R}^2 \mid t \in \mathbb{R}\}.$$

Appendix 2

Form used to measure motivation (items extracted from the Intrinsic Motivation Inventory, IMI)

Question 1. Indicate how true the following statements are for you using a scale of 1 to 5

(1 = nothing true, 2 = very little true, 3 = something true, 4 = true, 5 = completely true), or complete the sentence where ellipses appear.

1.1 I think that algebra may be of some value to me.

1.2 I think knowing algebra is useful for ...

1.3 I think knowing algebra is important because ...

1.4 I would be willing to study algebra again because it has some value for me.

1.5 I think knowing algebra could help me ...

1.6 I think knowing algebra could be beneficial for me.

1.7 I think knowing algebra is important.

1.8 I think algebra classes are interesting.

1.9 I think algebra exercises are entertaining.

1.10 I work hard to study algebra.

1.11 For me it is important to do well.

1.12 I am relaxed while working in algebra.

Question 2. On the scale of 1 to 5 (1 = none; 2 = low, 3 = neutral, 4 = quite a lot, 5 = a lot), rate the usefulness of an engineering student knowing the following concepts:

2.1 Vector space

2.2 Vector subspace

2.3 Space generated by one or more vectors

2.4 Parametric and Cartesian equations of subspaces

2.5 Vector linear independence and dependence

2.6 Basis of a vector space

2.7 Dimension of a vector space

2.8 Coordinates of a vector in a basis

Content knowledge question:

Question 3. Reason if the following statements are true or false. If true, it is necessary

Give a demonstration. If a statement is false, it is enough to set an example with numbers for which it is not met.

3.1. $S = \{(a + c, a - b, b + c, 0) \in \mathbb{R}^4, \text{ such that } a, b, c \in \mathbb{R}\}$ is a vector subspace of \mathbb{R}^4 .

3.2. $H = \{(x, y) \in \mathbb{R}^2, \text{ such that } x \geq 0, y \geq 0\}$ is a vector subspace of \mathbb{R}^2 .

Question 4. You can add any comments you deem appropriate to improve the theme of spaces vector or algebra subject.

Appendix 3

Question on vector spaces in the course exam

Note: vectors are indicated in bold letter.

Experimental group students' course exam:

1. Consider the vector space \mathbb{R}^3 and the subset $H = \{(x, y, z) \text{ in } \mathbb{R}^3 / z = 0\}$
 - 1.1. Prove that H is a vector subspace.
 - 1.2. Write parametric equations and a basis of H .
 - 1.3. Does the vector $\mathbf{v} = (2, 0, 4)$ belongs to H ? Reason the answer. If it belongs to, find the coordinates of \mathbf{v} in the base calculated in 1.2.
 - 1.4. Does the vector $\mathbf{w} = (4, 2, 0)$ belong to H ? Reason the answer. If it belongs to, find the coordinates of \mathbf{w} in the base calculated in 1.2.
 - 1.5. Are vectors \mathbf{v} and \mathbf{w} linearly independent? Reason the answer.
 - 1.6. Complete the basis found in the first section to a basis of \mathbb{R}^3 .

Control group students' course exam:

1. Given the subspaces of \mathbb{R}^4 , $U_1 = \{(x, y, z, t) \in \mathbb{R}^4 / x + t = 0\}$, $U_2 = \{(x, y, z, t) \in \mathbb{R}^4 / x = t = 0; y + z = 0\}$. If we consider the set formed by the vectors of the basis of the two subspaces U_1 and U_2 together, is this set a basis of \mathbb{R}^4 ?

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