Supporting Teachers in the Creation of Animations and Applets through Leveraging Content Knowledge

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Abstract: The integration of animations and applets in mathematics education has been widely recognized for its potential to enhance learning. Despite the growing number of free resources such as Desmos and GeoGebra, instructors often find that these tools do not fully meet their specific pedagogical needs. This manuscript explores the benefits of implementing a short unit focused on training teachers to create their own animations and applets. In particular, this study emphasizes the importance of Technological Content Knowledge (TCK) and the reciprocal relationship between an individual's content understanding and the capabilities of existing technology, namely Desmos Graphing Calculator. This study's findings demonstrate how engaging teachers with graphing technologies through their mathematical understandings can both deepen their understanding of mathematical concepts and enhance their ability to leverage technology in the classroom.

Introduction and Background Literature

The use of technology to improve mathematics education has long been a heavily researched area (Borba et al., 2017; Cullen et al., 2020; Jensen & Scott, 2022; Kaput & Thompson, 1994). Proper implementation of technology can lead to gains in higher-order thinking skills (Pujiastuti & Pambudi, 2025; Silverman, 2018; Wijers et al., 2010), support student achievement (Dunn & Kennedy, 2019; Mistretta, 2005), and improve teacher education (Bray & Tangney, 2017; Lee, 2021). Many studies indicate how the integration of specific technologies has improved student understanding of mathematics concepts across a variety of domains, such as Linear Algebra (Mauntel & Zandieh, 2024; Wawro et al., 2023), Calculus (Dilling & Witzke, 2020; Yu, 2023, 2025), Probabilistic Reasoning (Sullivan, 2022), and Geometry (Bhagat & Chang, 2015; Jablonski & Ludwig, 2023).

One research area of technology in mathematics education that has been gaining traction is investigating the impact of animations and applets in supporting student learning. The benefits of using animations and applets to support learning in Science, Technology, Engineering, and Mathematics (STEM) education have been well documented (Hidayat & Wardat, 2024; Rachmavita, 2020; Schnotz & Rasch, 2005). One argument for employing animations is based on the need to support students in constructing dynamic imagery of concepts that involve a process or variation, and the fact that static graphics do not adequately convey these ideas. Broadly speaking, the research comparing animations to static graphics indicated that animations led to increased learning outcomes (Berney & Bétrancourt, 2016; Ploetzner et al., 2020).

Regarding mathematics education, numerous researchers have demonstrated how animations can be utilized to support student thinking. One early work was Thompson and Thompson's (1996) usage of a racing animation

(Thompson, 1990) that depicted how a racer's distance and time covaried, which the instructor leveraged to support a student in understanding the concept of rate of change. Similarly, Kidron and Tall's (2015) employment of a Mathematica applet enhanced students' understanding of the limit concept and the idea of infinity as a process. Animations and applets have continued to be an area of interest in mathematics education, such as supporting productive understandings of rate of change (Yu, 2023), supporting students in making conjectures and argumentation (Lingefjärd, T., & Ghosh, 2022), and making conceptual sense of graphs (Paoletti et al., 2024).

While the literature on animations and applets can seem promising, one concern is how to support instructors in accessing or leveraging the appropriate ones that would supplement their unique ways of teaching. Despite the numerous free applets available on websites such as Desmos Classrooms and GeoGebra, these applets do not always allow instructors to use them exactly as they intend. Whether the interface is unfamiliar or the animation is missing something the instructor wanted, this mismatch between what is available versus what is desired is a significant issue that warrants further investigation. Often, existing technology has clear merits, but teachers sometimes struggle to view how it can be productively leveraged as a pedagogical tool (Lu et al., 2022; Okumuş et al., 2016). One reason for this mismatch is that animations and applets do not teach; instead, the designer has a conversation in mind that they want to facilitate with students. Therefore, if the background ideas and intent of the animation are not salient to another instructor, this can lead to unproductive uses.

One solution to this issue of utilizing other designers' works would be for instructors to create their own animations and applets. However, this is a non-trivial task, and little research exists in this area. Existing papers mostly report on supporting teachers' use of technology to create graphs or virtual learning objects, but these are primarily focused on teachers learning about the interface or thinking about the content knowledge (e.g., Akkaya, 2016; Brand, 2020; Heck et al., 2019; Thurm & Barzel, 2020). Therefore, this manuscript aims to illustrate how to support teachers in creating animations and applets that also leverage their content understanding. As such, this paper investigates the following research question: How does a lesson on Desmos animations support individuals in developing both their mathematical and technological content knowledge?

Regarding this study, considerable work has been conducted on the development and understanding of technology among pre-service and in-service teachers. One finding that is clear in the literature is that many teachers have much to learn about the effective integration of technology in the classroom (Francom, 2020; Mercader & Gairin, 2020) and that there is a distinct need for technology instruction for pre-service and in-service teachers (Gimbert & Zembal-Saul, 2002; Spiteri & Chang Rundgren, 2020; Tondeur et al., 2019). Furthermore, numerous studies have demonstrated the benefits of technology instruction for pre-service teachers, including improved teacher knowledge (Wilson et al., 2020), support for diverse learning styles (Shen, 2011), and increased student engagement (Bray & Tangney, 2016). It is important to note that most technology involves learning about each one's specific user interface, which can often be overwhelming and unintuitive. For example, many studies report that newer teachers state that they did not leverage new technology since they found it to be overly complex, unaligned with their

pedagogical style, or lacked proper support (Erduran & Ince, 2018; Makina & Kadzere, 2022; Stein et al., 2020). It would be ideal if mathematics instructors could leverage and deepen their content understanding (e.g., knowledge of equations of lines on the XY-plane) when utilizing technology. While the literature is clear that students may struggle with mathematical representations (e.g., Oehrtman et al., 2008; Paoletti et al., 2022), having future teachers improve both their mathematical content and technological skills simultaneously would be an efficient way to further their pedagogical capabilities. Therefore, I argue that teachers should be trained to effectively utilize graphing technology such as Desmos, GeoGebra, and Graphing Calculator, where most input is done through mathematical representations.

Theoretical Background

In considering what a teacher needs to know to integrate technology into their classrooms meaningfully, Mishra and Koehler (2006) propose the Technological Pedagogical Content Knowledge (TPCK) framework that emphasizes the intersection of Content Knowledge, Pedagogical Knowledge, and Technological Knowledge. While the entire framework is significant to this area of research, for brevity's sake, the most relevant portion of this framework to this paper is Technological Content Knowledge (TCK), which refers to the reciprocal relationship between Content Knowledge (CK) and Technological Knowledge (TK). TCK involves considering the content material and how it may be represented and changed through technology. I illustrate this reciprocal relationship in this paper by describing teachers' understanding of mathematical notation and Desmos Graphing Calculator. Throughout the study, the teachers leverage and strengthen their CK (graphical representations) to build animations while reflecting on how to use the technology to represent the mathematics to their students.

The TPCK framework has been productive in teacher preparation research for examining teachers' obstacles to integrating effective technology in the classroom. Some studies explored the connection between teachers' TPCK and their beliefs about technology (e.g., Kimmons et al., 2020; Karakus, 2018; Valtonen et al., 2006). For example, Valtonen et al. (2006) reported that their teachers consistently chose technology that aligned with their pedagogical beliefs, even if they were aware of existing technology that supported a learner-centered approach. Many of these studies observed that although these teachers had furthered their Technological Knowledge, they did not leverage these skills in their teaching practices (Voogt et al., 2013). One implication from this literature is that even when teachers have gained knowledge about how to use certain technologies, it will remain likely unused if it does not support them in having the types of conversations they want to have with their students. Therefore, supporting teachers in building their own animations and applets would likely circumvent this issue, as they would be the ones to create what they would need for their classes. With regard to mathematics education, other studies have leveraged TPCK to describe productive teacher preparation and professional development (Kartal & Çınar, 2024; Niess et al., 2009; Weigand et al., 2024).

When examining the TK of teachers, many researchers refer to TK as procedural knowledge of how to use specific technology, such as "knowing how to operate a computer and knowing how to use a multitude of tools/software as

well as troubleshoot in problematic situations" (Angeli & Valanides, 2009, p.158). These studies focus on developing teachers' procedural fluency so that they can effectively employ technology. While these works have demonstrated gains in teachers' TK, this manuscript provides an example of mitigating the hurdle of learning an esoteric interface by using a technology (Desmos Graphing Calculator) that primarily uses the teachers' content knowledge. For example, instead of using an interface to create different frames for an animation manually, a teacher could leverage Desmos' slider functionality and consider what aspect of a graph or object they want to vary. Instruction focused on this type of technology would simultaneously support a teacher's content and technological knowledge. Therefore, this paper provides an example of how leveraging technology that aligns with the content domain can be effective in teacher preparation.

Animations as Didactic Objects

When teachers employ technology in the classroom, they generally consider how it impacts learning (or what Mishra and Koehler term Technological Pedagogical Knowledge), including the potential discussions they wish to have with their students. This is similar to what Thompson (2002) refers to as a didactic object in that the technology itself does not do the teaching. Instead, the instructor has in mind a mathematical conversation they would like to facilitate with that technology. In this context, "mathematical conversation" refers to how a teacher envisions the discussion and activities in the classroom might happen based on some particular mathematical idea. These conversations may occur during a classroom discussion, in small groups, or as students interact with the didactic object. Therefore, it becomes apparent that a teacher having the ability to modify existing animations or create entirely new ones would be a desirable skill to facilitate the kinds of mathematical conversations they envision. For example, suppose a teacher wanted to illustrate exponential growth with attention to how a 1-unit change in the input gives an associated multiplicative change, no matter where the 1-unit change occurs. While there are online resources for exponential growth animations, they may not have the exact functionality the instructor desires, such as marking off 1-unit changes or only showing certain portions of the graph. Even if this teacher wanted to change existing animations, most online applets are esoterically designed and provide little to no assistance in modifying them.

In order to effectively use graphing calculator technologies such as Desmos, instructors need to leverage their mathematical understanding of graphs and variables to create dynamic illustrations. To do this, I employ quantitative reasoning (Smith & Thompson, 2007), which involves analyzing a situation in terms of quantities and the relationships among them. In this context, I focus on supporting teachers in reasoning about what they are trying to represent with mathematical symbols (quantification and symbolization) and considering what quantity's value they are trying to vary when creating an animation. For example, suppose an individual wants to create a vertical line; this is generally of the form $x = a$ where a is a constant. Now, suppose they want to make it a line segment. In this case, they could consider what quality of the line they are trying to constrain, which might lead them to determine that they want to impose a restriction on the y values, such as $0 \leq y \leq 4$. It is worth highlighting that this is not something particular to a technology's interface; instead, it leverages an individual's mathematical understanding.

Methodology

This study was conducted through several in-class implementations with pre-service teachers and mathematics graduate students (in separate classes). The former involved 2 classes of a Technology in Teaching Mathematics course, while the latter involved 3 cohorts of a teaching professional development program for mathematics graduate students. In both cases, the content was mostly the same, with the undergraduate course given more time to work through the material. In these classes, students worked through a pre-made Desmos Classroom, where they were shown samples of animations to build. The necessary concepts to build these animations involved content ranging from Algebra to Pre-Calculus, which all of these pre-service teachers and graduate students had all taken at some point in their academic career.

During the first session, the instructor facilitated a conversation to engage them in mathematical thinking about what they were trying to represent, what qualities of the situation were varying, and how to symbolize it through mathematical notation. Afterward, students were tasked with replicating different animations from scratch. Students were initially asked to replicate rather than create their own animation, as this allowed the instructor to facilitate a conversation around the same mathematical concepts, reduce differences in approaches, and train the students in using Desmos effectively. After completing these introductory courses, students were tasked with creating their own animations tailored to their specific needs and interests. Compared to existing studies (e.g., Akkaya, 2016) that focus on TK or CK separately, this study presents a unique setting where the individuals simultaneously engage and develop both knowledge bases.

To track the students' development, data was collected in the form of their responses to tasks throughout the activities, their equations on the Desmos pages, audio recordings of the classes, and responses from an exit ticket at the end of the last session. During each class, the instructor began by encouraging students to think aloud while working through tasks, in order to capture their thinking through the audio recordings. The audio recordings were then transcribed and cleaned prior to analysis. Students' work in the Desmos pages was done through Desmos Classroom, where the instructor periodically screen-captured students' work from the teacher's side of the activity to record students' various attempts throughout each task. Additionally, the instructor made notes of these screenshots as they made them to integrate them with the audio transcripts better. The exit ticket at the end of the last session consisted of 2-3 short, free-response questions (depending on whether it was for the pre-service teachers or the graduate students) about their experiences with the tasks.

An entire round of data was collected each semester from each group of students, resulting in a total of five subsets of data. Each round of data was then compiled to track each group's progress by matching up their work on the Desmos classroom pages and their corresponding transcripts from the audio recordings. Exit tickets were not included in this compilation, as they were anonymous surveys, and instead, they were compiled across all five rounds of data to identify recurring themes.

The data analysis of the audio transcripts was conducted through the lens of quantitative reasoning (Smith & Thompson, 2007) by examining how the students reasoned about what quality or aspect of the situation they imagined as varying and how they attempted to quantify or symbolize that varying quantity. This is what Carlson et al. (2023) term *emergent symbol meaning*, which is an important perspective to note because symbols and notation do not inherently carry meaning; instead, a student associates mathematical meaning to specific notation based on their experiences with mathematics. Using quantitative reasoning as a theoretical framing, I analyzed the audio transcripts using Open and Axial Coding (Strauss & Corbin, 1990) for moment-by-moment coding of students' responses and interpretation. Using the codes from each group, a thematic analysis (Clarke & Braun, 2013) was conducted within and across each group to identify patterns in how the students approached the tasks.

Graphing Dynamic Lines

Figure 1

Snapshot Of An Animation Of A Linear Function In Desmos
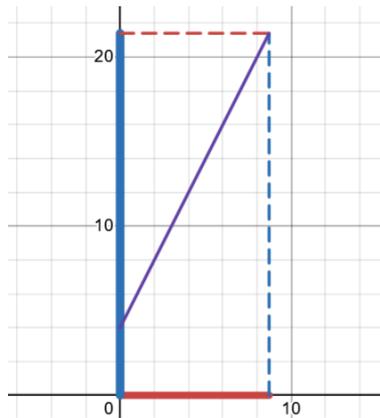
In the first class, the instructor facilitated a mathematical discussion to model the thinking the instructor wanted the students to engage in throughout the lesson. In this initial task, the class worked on creating a graph of a linear function that would emerge as a result of coordinating the input and output quantities simultaneously (Figure 1). This was chosen as the first task because it kept the cognitive load fairly low, allowing the instructor to focus the conversation on equations, the domain of a function, and how to use Desmos. Students chose a generic linear function, such as $f(x) = 2x + 4$, and the instructor guided the students in creating an animation that would show the trace of the graph as the input variable increased. To achieve this, the instructor focused the conversation on mathematical representations in a graphical setting and subsequently on the quality of that representation that would vary. For example, when creating the solid horizontal red line in Figure 1, the instructor asked, "How do we represent the graph of a horizontal line with math notation?" After the students determined it would be of the form $y = c$, the instructor referred to the animation and asked, "What about this horizontal line is varying in this animation, and what would that look like in mathematical form?" The students discussed it in small groups and came to an agreement that it involved limiting the domain, which entailed adding the restriction $\{0 \leq x \leq t\}$ to the function definition (Figure 2). This

example represents the kind of mathematical engagement and reflection the students consistently engaged with, in that what they needed to type in was not some esoteric code. Instead, they thought about the mathematics they knew and typed that into Desmos' interface.

After the graphing dynamic lines task, students were shown an animation and worked in groups of 2-4 students to recreate those animations in Desmos. For example, in the following task, the students needed to create a similar animation for an exponential function (Figure 3). While the instructor periodically went around to each group to listen to their conversation and assist the students when needed, the students mainly discussed their work within their groups. The following section provides excerpts of how students simultaneously engaged in quantitative reasoning and utilized the Desmos technology.

Figure 2

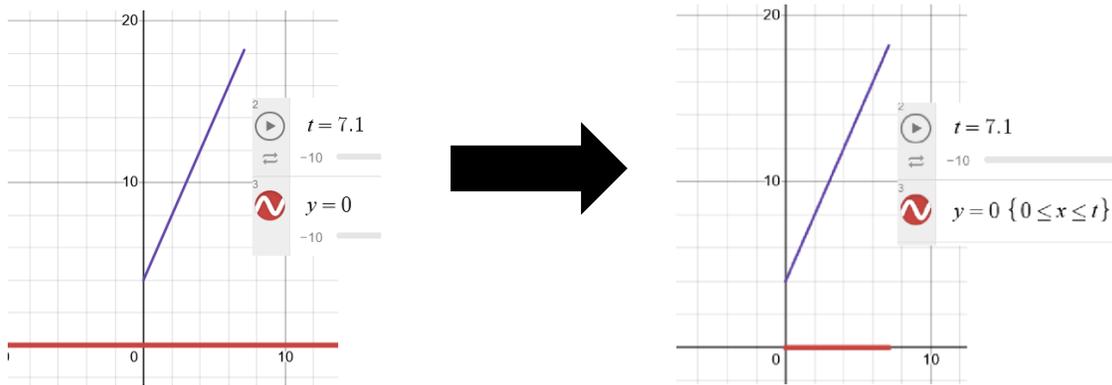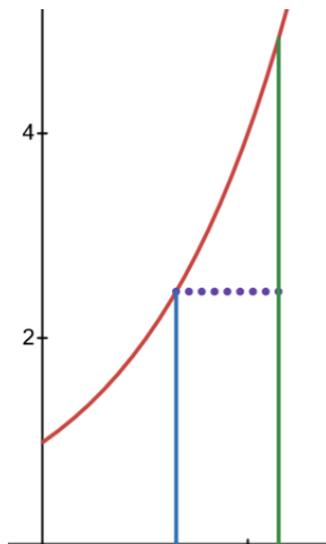*Students Employing Mathematical Symbols To Impute A Domain On The Horizontal Line*

Figure 3

Snapshot Of The Exponential Function Animation

Results

In the Exponential Function Animation Task, students were asked to create an animation for the function, $f(x) = 2^x$, which would show that for any 1-unit change from the input, $x = t$, the associated value at $f(t + 1)$ is always twice as large as $f(t)$. The animation would involve a slider for the parameter t, illustrating this property of exponential functions as the input value varies smoothly and continuously (Exponential Function Animation).

Table 1

Three Students Creating The Vertical Lines Of The Exponential Function Animation

1	Stu 1	So like we want the line to show the y-value at the current input, how to make it
2		move again?
3	Stu 2	Like with the slider, we can use t *types in $y = t$*
4	Stu 3	Ah um, it made a horizontal line, we want a vertical line so that's $x = t$
5	Stu 2	Don't we want it to go up where the graph is there? Like stop drawing that line
6		when it hits it?
7	Stu 1	Oh yeah, we did that in the last problem right…how did we do that again?
8	Stu 3	We did the brackety thing
9	Stu 1	What?
10	Stu 3	We need to put umm the values we want to graph inside brackets
11	Stu 2	Like this? *types in $\{0 \leq x \leq f(t)\}$*
12	Stu 3	Yeah something like that, but nothing changed on the graph
13	Stu 1	Oh, well I think what we wrote is referring to the x values, but our line is going
14		up and down up to that point… so like we need to tell it about the y-values
15	Stu 2	Ah change this thing, the x to a y right? *types in $0 \leq y \leq f(t)\}$*

Table 1 provides an account of a group of 3 pre-service teachers who worked through creating the vertical blue line in Figure 3. In this excerpt, the three students engaged with representing a vertical line and limiting its range using mathematical notation. Initially, the students conflated making a vertical line with an equation involving $y = $, likely because they wanted to "show the y-value at the current input" [Lines 1-4]. Afterward, the students discussed limiting its range to have the line stop where they intended. At first, they used informal language such as "brackety thing," and they eventually referred to the notation to make sense of why the graph did not display what they intended [Lines 7-15]. Throughout this excerpt, the functionality of Desmos in displaying the students' mathematics supported them in making shifts in their symbolization and making sense of those symbols. For example, when the students initially typed in $\{0 \leq x \leq f(t)\}$, they noticed that nothing changed on their graph, which led them to realize that their symbolization was "referring to the x-values" and that they needed to "tell it about the y-values" since the line was "going up and down" [Lines 12-15]. Quantitatively speaking, Student 1 here made a realization that this inequality, $0 \leq x \leq f(t)$, referred to a restriction on their input variable, and that since the line they were imagining was moving up and down, that meant that "we need to tell it about the y-values" [Line 14]. Here, the students shifted their understanding by making quantitative sense of this inequality as restricting the potential values of a variable. This shift is evidenced by Student 2's immediate response of "ah change this thing, the x to a y right?", where they recognized what needed to be changed in their statement. Additionally, throughout this whole passage, the students

were not guessing at how to correct their mathematical representation; instead, they reflected on what their symbols meant and made accommodations by considering what quality of the line they were trying to impact.

Table 2

Two Students Creating The Horizontal Line In The Exponential Function Animation

1	Stu 4	Um the purple line, how do we get that to be right there?
2	Stu 5	Something like $y =$ right?
3	Stu 4	Oh, but… equal to what?... It should be the y-value, but $y = y$ doesn't sound
4		right
5	Stu 5	Hold on lets go back and look at it (the sample animation) again
6		…
7	Stu 4	Oh, it err moves with the other lines… it's always on top of the left one
8	Stu 5	Yeah… oh it's the y-value for that x-value there
9	Stu 4	So like $f(x)$? *types in $y = f(x)$*… that didn't work… what did we do wrong
10	Stu 5	It moves, like x is like all the values I guess, we need to tell it for whatever that
11		value is where we drew the vertical line, that's the t over there. *types in $y =$
12		$f(t)$*

Table 2 involves an excerpt of a pair of pre-service teachers who engaged similarly to the first group when creating the dashed horizontal purple line in Figure 3. Initially, the students struggled to distinguish between their understanding of y as a variable versus the y-value at a particular x-value [Lines 2-9]. However, having the sample animation [Lines 5-7] and an environment to test their ideas [Line 9] allowed them to simultaneously engage in understanding mathematical notation and the associated graphical representations. Student 4's initial thought of wanting to obtain the output of the function at the current x-value by typing in "$y = f(x)$" and Desmos not showing what they expected allowed the pair to have a discussion centered around the mathematics. Student 5 articulated their realization by saying, "Like x is all the values I guess" and subsequently realized that the current value of x was represented by their t-slider [Lines 10-12]. In this excerpt, Students 4 and 5 realized how functions use the input variable, x, as something that takes on multiple values [Line 10], and that to call on a specific value as that variable varied, required the use of the parameter, t. This excerpt demonstrates how building this animation supported them in making a more profound sense of the role of variables versus a specific value of that variable.

In the Water Tank Animation Task, students were tasked to create an animation that would illustrate a cylindrical water tank filling up with water (Figure 4) [Water Tank Task]. To create the illusion of water filling up the tank, students needed to create an outline of an ellipse and utilize an inequality to shade the set of all points between the top half of the ellipse and the bottom portion of the tank. Initially, the instructor led a short discussion on what "shading" would be in a graphical setting, to which the students recognized that it would involve inequalities. After setting up the initial shape of the cylinder by using the equations of ellipses, the students worked in groups to create the animated portion of the task. For reference, the students made a cylinder that had a radius of 4 and a height of 10 with the center of the base situated at $(0,0)$; therefore, the equations of the associated ellipses were $\frac{x^2}{16} + y^2 = 1$ and $\frac{x^2}{16} + (y - 10)^2 = 1$.

Figure 4

Snapshot Of The Water Tank Animation Task
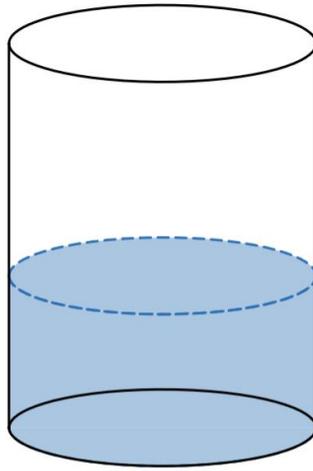
Figure 5

Depiction Of How Some Groups Approached The Water Tank Animation
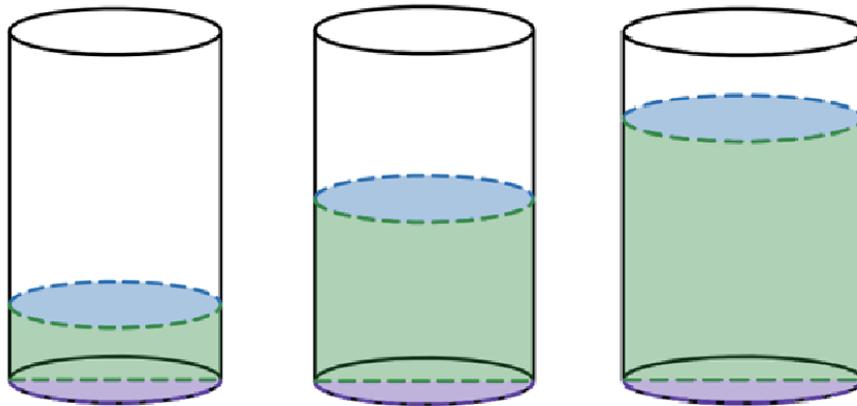
Each of the groups approached the task slightly differently. For instance, several groups divided the area to be shaded into three corresponding regions. Figure 5 illustrates the most common approach these groups took to complete the task, with minor variations in how the three pieces were divided.

One group started by creating the bottom purple section in Figure 5 (the static inequality, $0 \leq y \leq -\sqrt{1 - \frac{x^2}{16}}$) but had trouble when they wanted to create the top blue ellipse that would move upward. This group initially tried $\frac{x^2}{16} + y^2 + t \leq 1$, where t was the parameter of their animation slider, where they likely were thinking of t as a value added to the y-value that would vertically translate the ellipse. This is understandable since much work on translating graphs involves functions (rather than conics), where adding a constant value would translate the base graph vertically.

When the animation did not move according to their expectation, the group had a discussion and eventually asked the instructor for some assistance [Table 3]. Initially, the group had the right idea of trying to impact the y-value to "move it upward" [Line 14], but noticed that their attempts to add a constant value (the parameter, t) did not correspond to that motion. As a clarification, the students here were not guessing as to where to put the t. Instead, they were still operating under the idea that adding a constant value equates to vertical shifts; therefore, it made sense to them to try adding it to the other side of the inequality. When asked, the instructor guided the students to consider what aspect or attribute they were trying to vary [Lines 11-12] and to utilize their understanding of how the ellipses were created, in particular, the one that had the center that was 10 units directly above the origin [Lines 15-18].

Table 3

Students Attempting To Create The Moving Blue Ellipse

1	Stu 6	Wait why is the circle just staying at the bottom… and also shrinking? Er… did
2		we type something wrong?
3	Stu 7	Maybe try adding the t to the other side?
4		(After some mild chatting, they type in $\frac{x^2}{16} + y^2 \leq 1 + t$)
5	Stu 6	Now its expanding and not moving up…. At least its animated growing
6		*laughs*… [Instructor's name], we're having some trouble
7	Ins	Yes?
8	Stu 6	We can't get our circle to go up…
9		(Group plays the animation for the instructor)
10	Ins	Okay so tell me what you were trying to do with this line over here
11	Stu 6	Uh yeah, we wanted to move it up, so we thought adding the t would do that…
12	Ins	Okay, can you say a little more about that? Like what aspect were you trying to
13		vary there?
14	Stu 7	The y value, like that's what would move it upward right?
15	Ins	I see, you've got the right idea…um we have to think about how this equation
16		works… like how did you all create the ellipse uh up here? I think that'll help
17		you get what you want, remember think about what quality of the object you
18		want to vary and then connect that with how that is represented.

After some discussion, the two students manipulated their upper ellipse equation, $\frac{x^2}{16} + (y-10)^2 = 1$, by choosing different values for the 10 and trying out different coefficients to see what would appear in the Desmos applet. They realized that in the general equation for an ellipse, $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, the k impacted the y-coordinate of the center of the ellipse, and they used that knowledge to create the moving top blue ellipse in Figure 5 via $\frac{x^2}{16} + (y-t)^2 = 1$. At one point, one of the students stated, "ah, the thing moves up because the part we added (t) is moving the center of the circle like vertically changing." This student's statement evidenced that they were not only successful in creating the animation but also fruitful in strengthening their understanding of conics. In particular, these students transitioned from an understanding of "adding a constant value will move the graph upwards" to one that entailed how varying different features of the equation would impact its graph. This outcome likely occurred due to Desmos' graphing capacities to show students the result of their mathematical symbolizations and the instructor's suggestion to reason about quantities. For example, while the students did not immediately know how to manipulate their equation of the

ellipse to move it upwards, they leveraged Desmos by testing out different values and then generalized from that situation.

In the class with the graduate students, one pair approached the task by creating a combination of the green and blue sections of Figure 5. Initially, these students were successful in mathematically symbolizing the moving blue ellipse and subsequently identified the top half of that ellipse as $y = \sqrt{1 - \frac{x^2}{16}} + t$. They then wanted to fill in the area underneath that curve to the x-axis, which led to a productive conversation about the mathematics and the associated graphical representation (Table 4). This pair started with what they recalled as the definite integral to represent the "area under the curve" [Lines 1-3]. When what they expected to show up in their animation did not happen [Lines 4-7], they began trying other ideas while also making sense of the mathematical representations. When they tried $y < \int \sqrt{1 - \frac{x^2}{16}} + t \, dx$, they noticed how the area under $y = 28.6$ was shaded in, prompting them to reflect on their symbolization [Line 10]. This reflection led to one of the students articulating that the inequality involved the set of all points that would make that inequality true [Line 11] and that the integral represented the numerical value of the area, not the set of points that composed that area [Lines 12-15]. Soon afterward, this group programmed the area they intended, based on their understanding of how Desmos functioned.

Table 4

Two Graduate Students Making Sense Of Mathematical Representations

1	GS 1	We gotta have it filled under that parabola ($y = \sqrt{1 - \frac{x^2}{16}} + t$), isn't that just the
2		integral?
3	GS 2	uh…ah you mean like the area under the curve?
4	GS 1	Yeah let's type that ($\int \sqrt{1 - \frac{x^2}{16}} + t \, dx$)… nothing
5		(About a minute of checking if they typed things in correctly)
6	GS 1	Aren't we trying to get this area here?
7	GS 2	Yep… I mean we have what that area is but it's not showing up
8	GS 1	Do we need to do something to the integral then?
9		(The students test out various manipulations such as $y < \int \sqrt{1 - \frac{x^2}{16}} + t \, dx$)
10	GS 1	Wait, so why does this one now have things shaded but like everywhere?
11	GS 2	Umm … well this inequality is saying like what y-values are less than that
12		integral… but that integral is the area…
13	GS 1	Isn't that what we want?
14	GS 2	No no no no, wait. The integral… is the area… like right now that integral is
15		28.6… so it's going to shade below $y = 28.6$

In this excerpt, the two graduate students had a nuanced shift in their understanding of the definite integral and its graphical representation. Initially, they recalled it as "the area under the curve" and understandably attempted to use that to shade in the corresponding area. However, by having Desmos immediately display (or not display) their mathematics, this caused the pair to reflect on their understandings and try out different versions of the integral. After

seeing how Desmos displayed the shaded area, the graduate students paused and realized the nature of their error: "But that integral is the area," and "Right now that integral is 28.6." In the end, these students realized that the definite integral represented the numerical value of the area, rather than the area itself.

Later in the class, this pair of graduate students noted that "Desmos is just graphing the set that we tell it," which evidenced their understanding that the lines and areas Desmos displayed were the set of all (x, y) pairs that made their Desmos equations true. In this excerpt, the ability to simultaneously engage with the mathematics and the Desmos interface catalyzed a conversation that deepened their understanding of mathematical concepts (Content Knowledge) and how to use Desmos to represent them (Technological Knowledge).

Post Animation Lesson Results – Sample Final Projects

At the end of the semester, both the pre-service teachers and graduate students had the option of creating an animation or applet as their final project. Many students utilized Desmos and created applets such as a sine function applet that allowed students to test different parameter values and an animation for optimization problems in Calculus (Figure 6). These products indicate that the students leveraged what they learned in these sessions and demonstrated a level of sophistication beyond what was taught in the technology lesson. For example, one graduate student teaching a section of Calculus created a visualization for optimizing the area of a trough by animating how the angle between the base and the sides would vary (Figure 7). As a note, this animation involved varying the slope of two lines as the angle varied, which was not directly covered in the professional development. This example, as well as the other animations students created, evidence that the students could generalize the process to make the animations by leveraging their mathematical understandings. Lastly, several graduate students later reported that when they used their animations in class, they felt that their class was more engaged and seemed to understand the content better than in previous semesters.

Figure 6

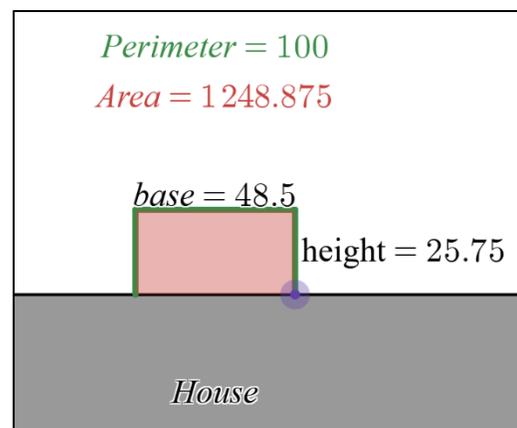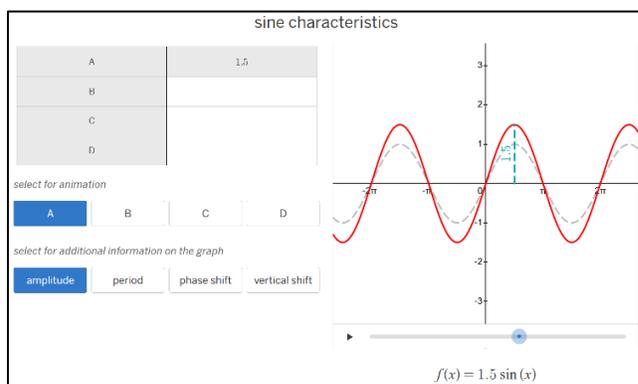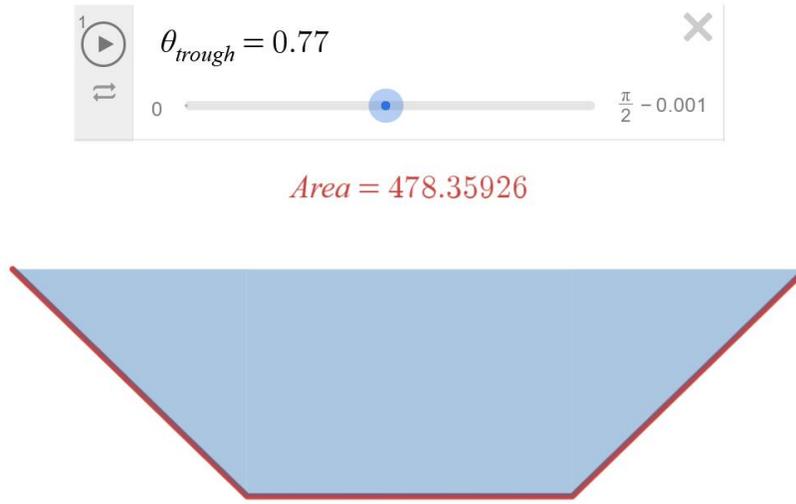*Samples Of Student-Made Animations*

Figure 7

Animation Depicting The Optimization Of The Area Of A Trough

While the content of these animations is not original or different from other existing animations, the main difference lies in how the animation was designed to serve as part of the teacher's idea of how the conversation would unfold (*didactic objects*). For example, as part of the final project, the students gave a brief presentation on the animation and demonstrated how it would be used in the classroom. The pre-service teacher who created the sine animation in Figure 6 discussed how they wanted to easily transition between transformations of the sine function based on its amplitude, phase shift, period, and vertical shift (in their animation, this is achieved by clicking on the associated feature). Additionally, they mentioned how they wanted to engage with their students by using the animation to hold a short session where the students would try to predict what would happen if they changed one of the A, B, C, or D values. While sine animations do already exist, this student created one that would supplement the mathematical conversation they wanted to hold for their students, rather than viewing the animation as the sole teaching agent.

At the end of the sessions, both groups of students answered questions about their experiences with their respective classes. One of the pre-service teachers stated that

> I mostly struggled with what objects were moving and in what way they were moving. For example, I struggled with what variables needed to be sliders and how I could change the animation to work for the task. I definitely accomplished a more in-depth understanding of different variables and their deeper meaning in connection with Desmos and making animations.

This student's reflection aligns with the findings of this study, as they described reasoning about how the qualities of an object were varying and that they were both engaging with and strengthening their mathematical understandings. In their final sentence, the student also mentions the reciprocal relationship between their CK ("understanding of different variables") and their TK ("deeper meaning in connection with Desmos"), noting that the lessons simultaneously bolstered their understandings of both.

Another pre-service teacher stated they "liked Desmos over the others (the other technology they used in the class) because it's straightforward, like I just needed to type in math and not have to figure out what button to press." Many other students shared this sentiment, demonstrating the benefit of using a technology aligned with their mathematical background. These students articulated that learning Desmos was preferred because they "just needed to type in math" as opposed to learning a brand-new software interface.

The graduate students also made similar remarks on their exit tickets. Even more noteworthy is that at the end of the graduate course (which involved other pedagogical topics), 80% of the graduate students indicated that the Desmos lessons were the most memorable aspect of the course. For example, one student stated, "The Desmos lessons were my favorite part of the class. I liked how concrete the examples were, and it was fun to learn about what is possible to do with Desmos. I enjoyed using what I learned in class to make my active learning project". Another graduate student wrote, "I found the Desmos part helpful because I was able to make something for my class that I was teaching that semester, so it was really practical and useful." These graduate students consistently indicated that they improved their TK ("what is possible to with Desmos"). Further, unlike the situation Voogt et al. (2013) noted about teachers developing their TK but not integrating those technologies into their classroom, these graduate students directly stated how much they appreciated the Desmos lessons because they could and eventually did incorporate them into their teaching.

Conclusion

The feedback and products from these students indicate noticeable benefits to supporting pre-service teachers and graduate students in technology lessons that simultaneously develop their content and technological understandings. By removing the obstacle of learning esoteric interfaces, these students demonstrated gains in their Technological and Content Knowledge. In summary, the benefits of employing such a lesson were (a) increased mathematical understandings of graphical representations and variables and (b) increased capacity to build one's own animations/applets.

With respect to mathematics education, this study indicates the benefits of training mathematics instructors in using Desmos (or similar technology such as GeoGebra) to expand their pedagogical capacities. Desmos' functionality of using typical mathematical notation to generate graphs allowed the students to deepen their content understanding while simultaneously developing their ability to create applets and animations. As the students worked through the tasks, their own meanings emerged for how they would represent certain mathematical concepts graphically. Moreover, when what they attempted to represent did not appear or work in constructing the animation, they were able to reflect on their emergent symbol meaning and made adjustments in their ways of thinking. It should be clarified that the medium itself is not solely responsible for doing this; instead, the instructor facilitated conversations that encouraged students to think in a way that catalyzed these opportunities.

This study demonstrates the benefit of engaging both an individual's content and technological understandings simultaneously. In this case, an instructor who profoundly understands how Desmos works (and the associated content knowledge) has the potential to generate multiple animations and applets across a variety of courses. For example, this study's content and tasks were primarily focused on Pre-Calculus topics. However, the students' final project products involved topics outside of Pre-Calculus, such as Optimization problems in Calculus (Figures 6 & 7). Additionally, the graduate students reported anecdotal improvements in their class due to their increased understanding of the technology, indicating the potential for such professional development to create shifts in the teaching of mathematics. To be clear, this paper does not argue against learning other technologies, as they offer additional affordances that Desmos does not. For instance, Geometer's Sketchpad provides a more substantial medium for teaching Geometry, whereas Desmos lacks dynamic drawing features. While it would be ideal if teachers had the opportunity to become skilled in multiple technologies, the time commitment and other constraints make a compelling argument for efficient technology training that can be applied to various contexts. This study demonstrates how training teachers on Desmos can productively enhance teachers' pedagogical capacities within a short time period.

As the mathematics education research field continues to learn more about supporting pre-service and in-service teachers effectively, I argue for more studies and professional development programs that support teachers in creating animations and applets. This is needed due to researchers' consistent calls for improving students' visualization of mathematics, which entails the employment of dynamic imagery (Presmeg, 2006; Thompson et al., 2013; Thompson & Harel, 2021). Therefore, enabling teachers to utilize animations and applets can promote students' understanding of mathematical concepts.

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