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Refining Progressions and Tasks: A Case Study of Design Cycles to Support Covariational Reasoning

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Abstract: Design-based research is a common tool mathematics educators use to study student learning and to generate high-level learning progressions and sequences of concrete mathematical tasks through iterative research cycles. There is a need for more transparent accounts of how researchers make decisions during the generation of such progressions and tasks. We address this need by describing the results of a case study of our design decisions, leveraging the constructs of local instruction theories and hypothetical learning trajectories to frame our decisions to promote students' quantitative and covariational reasoning. We describe four considerations that influenced our re-design both of mathematical tasks and of learning progressions to support students' covariational reasoning across seven teaching cycles with middle school students in the U.S. (ages 12-14). The four considerations that repeatedly influenced our task design decisions between cycles are: 1) supporting student thinking towards our goals, 2) eliciting student thinking, 3) keeping instruction efficient, and 4) exploring new possibilities. We discuss the importance of these considerations in our own work. We also highlight ways this work informs research on students' quantitative and covariational reasoning and provide implications for task design. Through this report, we intend to provide an account that examines the task design process with reflexivity in a way that is useful for other researchers.

Keywords: Local Instruction Theory, Hypothetical Learning Trajectory, Task Design, Quantitative Reasoning, Covariational Reasoning.
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Introduction

Design-based research (DBR) strategies have been commonly applied in mathematics education to investigate and generate learning progressions and tasks to support students' mathematical learning (e.g., Abrahamson, 2009; Cobb, McClain, & Gravemeijer, 2003; Gravemeijer, 2004; Larsen & Lockwood, 2013) At its core, DBR is a methodological approach that involves iterative cycles of creating, testing, and refining interventions to address real-world educational problems (Anderson & Shattuck, 2012; Design-Based Research Collective, 2003; Kelly, 2004). Two central products mathematics education researchers who use DBR to study student learning may generate are 1) high-level learning progressions and 2) sequences of concrete mathematical tasks. Importantly, we conceive of the relationship between progressions and tasks as dialectical within a design cycle.

We explain this relationship in terms of local instruction theories (LITs; Gravemeijer, 2004) and hypothetical learning trajectories (HLTs; Simon, 1995). Whereas a LIT is a conjectured course of reasoning through which students may engage with general supports to reach defined goals, an HLT includes specific actions, tasks, and tools to meet students' needs in specific learning context (Gravemeijer, 2004; Nickerson & Whitacre, 2010; Prediger et al., 2015). Nickerson and Whitacre (2010) summarize key differences between LITs and HLTs from Gravemeijer (1999) as follows: "HLTs are envisioned within the setting of a particular classroom, whereas an LIT comprises a framework, which informs the development of HLTs for particular instructional settings" (p. 228). Moreover, just as LITs can support the design of HLTs, implementations of an HLT can subsequently support the revision, expansion, or reimagination of the LIT itself (Nickerson & Whitacre, 2010). In this way, researchers can utilize both LITs and HLTs to refine both generalized learning progressions and sequences of concrete tasks over a design cycle, thereby creating useful tools for researchers and practitioners alike.

Despite the centrality of iterative processes to the design and refinement of LITs, HLTs, and other instructional products (e.g., tools and tasks), there is a need for increased transparency around how researchers engage reflexively in ongoing design decisions (Kieran et al., 2015; Sandoval, 2014). Kieran et al. (2015) describe that mathematics education researchers' clear connections between theory and task design within DBR "[remain] both underdeveloped and, even when somewhat developed, underreported" (p. 74). DBR practitioners have argued comprehensive accounts of ongoing design decisions can support more robust data analysis and further substantiate the significance of findings (Cobb, Confrey, et al., 2003; Edelson, 2002). As such, we address the need for increased reporting of the design process.

We present a case study (Stake, 1995) of our design decisions and revisions in a research project intended to support middle school students' covariational reasoning (or coordination of two changing quantities; Thompson & Carlson, 2017). The project consisted of seven teaching cycles where we tested instructional progressions and accompanying tasks with new groups of students. We introduce a LIT and initial HLT that informed our work, subsequently characterizing how we modified both products over time as we conducted and reflected upon our research. We highlight four considerations we observed as guiding our process: supporting student thinking towards the goals of our HLT and LIT (Consideration 1), eliciting students' thinking (Consideration 2), keeping instruction efficient towards our goals (Consideration 3), and exploring new possibilities (Consideration 4). By making public our own reflections, we aim to support researchers, both novice and expert, in examining new ways to integrate reflexivity into their task design processes (Lichtman, 2014; Tracy, 2010).

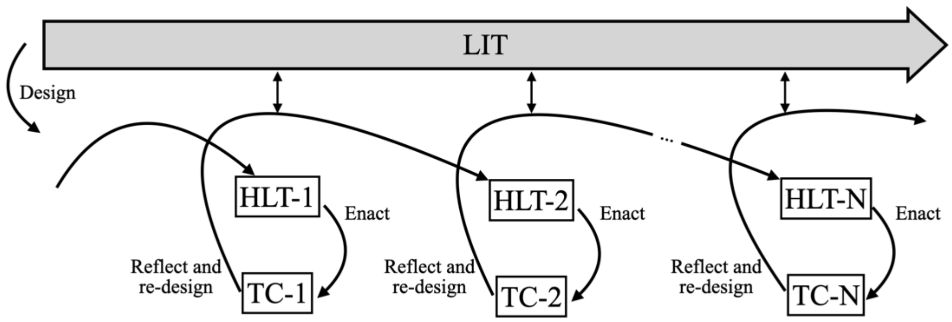
Below, we provide more details about LITs and HLTs in the DBR cycle. We then describe the constructs of quantitative and covariational reasoning relevant to the teaching cycles we analyze. Finally, we forefront the considerations as we refined tasks, HLTs, and LITs. Although the described process is situated in a particular context, we aim to be general enough that other researchers can glean valuable insights for their own work.

The Design Cycle: LITs, HLTs, and Teaching Cycles

At its core, we conceptualize the design cycle in line with Nickerson and Whitacre (2010): LITs anchor the design of HLTs, which are subsequently enacted through either small-group or whole-class teaching cycles. Following a teaching cycle, researchers may engage in a process of reflection and re-design of the HLT, enacting these new ideas again in the context of a new teaching cycle. Throughout the process, enactment can also inform re-design and reflection with respect to the larger LIT underpinning the project. As such, both LITs and HLTs undergo careful examination and refinement through a broader design process, which we illustrate in Figure 1.

Figure 1

A visual to connect iterations of teaching cycles (TC), HLTs, and LITs. Adapted from Nickerson and Whitacre (2010, p. 230)



Importantly, as instructional products, LITs and HLTs share several common and distinctive features that position them to mutually inform one another. For one, LITs and HLTs both include clear and connected goals, envisioned routes for student learning, and rationales, distinguished from one another in terms of their specificity (Gravemeijer, 2004; Nickerson & Whitacre, 2010; Simon, 1995). As Gravemeijer (2004) argues, HLTs are unique from LITs in that they account for “the planning of instructional activities in a given classroom on a day-to-day basis” (p. 107). That is, HLTs support the development of concrete tasks that can be implemented with particular students in mind (Simon & Tzur, 2004). Tasks may be designed with a context or situation with affordances for the envisioned learning route at hand (e.g., Ellis et al., 2012; Gravemeijer et al., 2003; Paoletti, 2019; Paoletti et al., 2023). Furthermore, researchers can consider supporting tools that may enable or constrain student activity (e.g., Gravemeijer, 2004). Importantly, task design decisions should align with and support the original goals and intentions of the HLT and LIT to ensure the construction of cohesive and substantiated products.

The ultimate goals of the design process described above involve generating instructional products that are useful to and adaptable by practitioners in context. Due to the complexities of studying real-world practice, it is important for researchers to articulate their primary focus versus what is ancillary when engaging in DBR (Cobb, Confrey, et al., 2003). However, this focus must also be balanced with researcher imagination. Letting student thinking take the lead can also illuminate new possibilities (Steffe & Thompson, 2000). Gravemeijer (2004) described the cyclic process of refining a LIT and HLT can “spark ideas that surpass what is tried out in the classroom” (p. 112), leading to new or extended progressions. Through a careful response to the pushes and pulls of the constraints and possibilities of practice, HLTs and LITs can be designed and refined in ways that impact teachers, instruction, and curriculum.

Quantitative and Covariational Reasoning

We explored how to support middle school students in constructing and reasoning about two quantities in dynamic situations (i.e., reasoning covariationally). Therefore, theories of quantitative and covariational reasoning were central to our iterative designs of LITs and HLTs. We adopt Steffe, Thompson, and colleagues' (e.g., Smith & Thompson, 2008; Steffe, 1991) stance on quantitative reasoning. From this perspective, an individual constructs a quantity when they conceive of a measurable attribute of some object or phenomena (Thompson & Carlson, 2017). This definition underscores that quantities are conceptual entities individuals construct to make sense of their world (von Glasersfeld, 1995). As such, teachers or researchers cannot assume students maintain meanings for quantities compatible with their intentions; students' constructions are consistent with their own conceptions but can be inconsistent with the quantity researchers intended for them to explore (Moore & Carlson, 2012; Paoletti, 2015; Paoletti et al., 2024). Given the potentially idiosyncratic nature of quantities, researchers must design tasks that ensure transparency with respect to the quantities with which students are reasoning.

When students coordinate two varying quantities, they engage in covariational reasoning (Thompson & Carlson, 2017). We leverage Carlson et al.'s (2002) covariational reasoning framework, which specified mental actions allowing for a fine-grained analysis of students' activity. These mental actions include coordinating directional change (e.g., as one quantity increases, the other also increases) and amounts of change (e.g., as one quantity changes by equally successive amounts, the other quantity changes by increasing amounts). By coordinating directional and amounts of change, middle school students have been able to describe and classify different types of relationships, such as those that are linear, quadratic, and exponential (e.g., Ellis et al., 2012; Fonger et al., 2020; Paoletti & Vishnubhotla, 2022).

We use an example from Stevens et al. (2015) to highlight affordances of reasoning with directional and amounts of change. Stevens et al. (2015) asked college students to graph the relationship between the height and surface area of a growing cone (see Figure 2). One student first considered the directional changes (i.e., as the cone's height increased, its surface area also increased). Then, the student drew consecutive cones and highlighted the increasing changes in surface area for equal changes in height (e.g., arguing the shaded area in the third cone in Figure **3**a was larger than the shaded area in the second cone). She then represented increasing amounts of change of surface area using increasing orange vertical segments on a coordinate system (Figure **3**b), ultimately graphing the covariational relationship. This student's contributions highlight how conceiving directional and amounts of change can support sophisticated covariational reasoning about relationships.

Figure 2

Screenshots of the Cone Task applet.

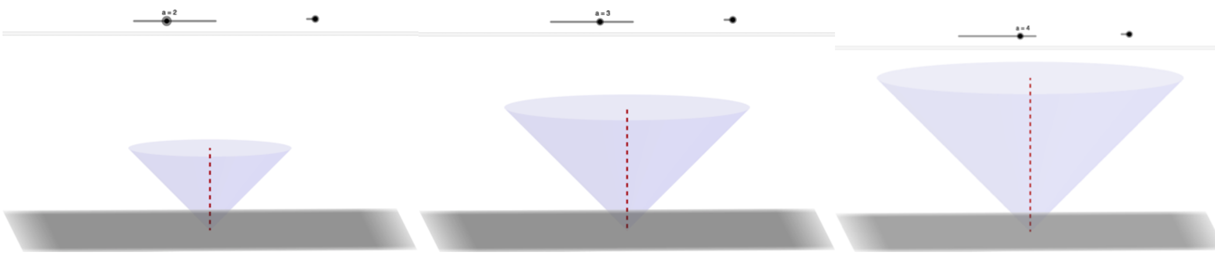
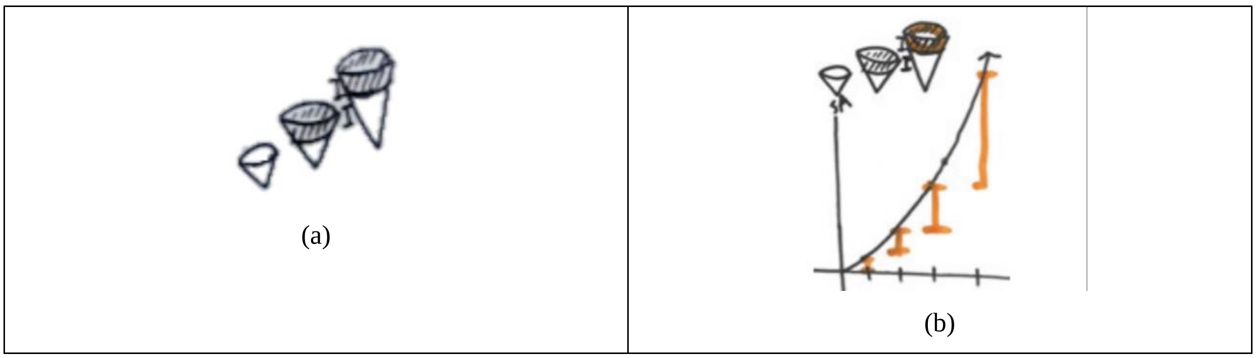


Figure 3.

Figures from Stevens et al. (2015, p. 368) showing a student's activity addressing the Cone Task.



(a) (b)

Our Study

One goal of our study was to create a LIT and an HLT that could support middle school students in constructing quantities and reasoning about the directional and amounts of change of these quantities.[1] Whereas researchers (Johnson, 2012; Moore, 2014; Moore et al., 2013; Paoletti & Moore, 2017) have described productive ways older students can engage in reasoning compatible with Carlson et al.'s (2002) characterizations, fewer studies have explored the possibility of middle school students enacting these mental actions (e.g., Ellis et al., 2012). Given that most school mathematics relationships exhibit either increasing, decreasing or constant amounts of change of one quantity with respect to the second, we considered this way of reasoning could be a foundation for middle school students' future understanding of mathematical relationships between quantities (e.g., Figure **3**b).

Table 1 presents our initial conjectured LIT and HLT relative to students' quantitative and covariational reasoning. For our LIT, we conjectured students should first construct two changing quantities in a dynamic situation (LIT_A).

[1] Our eventual goal was for students to leverage such reasoning to develop meanings for graphs. We refer the reader to Paoletti et al. (2022) and Paoletti et al. (2023) to explore how the tasks described in this paper supported students' graphing activity.

We hypothesized students could conceive of a directional relationship between two constructed quantities (LIT_B) before conceiving of the amounts of change of one quantity with respect to the second quantity (LIT_C).

Table 1

Our conjectured LIT and initial HLT

LIT	HLT-1
Goal – Students will be able to reason quantitatively and covariationally to conceive a relationship entailing both directional and amounts of change.	
LIT_A – Construct two changing quantities in a dynamic situation; LIT_B – Conceive a directional relationship between the two quantities constructed in LIT_A; LIT_C – Conceive of amounts of change of one quantity with respect to equal changes in the second quantity from LIT_B.	$HLT\text{-}1_A$ – Construct changing surface area and height of a dynamic cone; $HLT\text{-}1_B$ – Conceive that height and surface area both increase; $HLT\text{-}1_C$ – Conceive that for equal changes in height, the amounts of change of surface area increase.

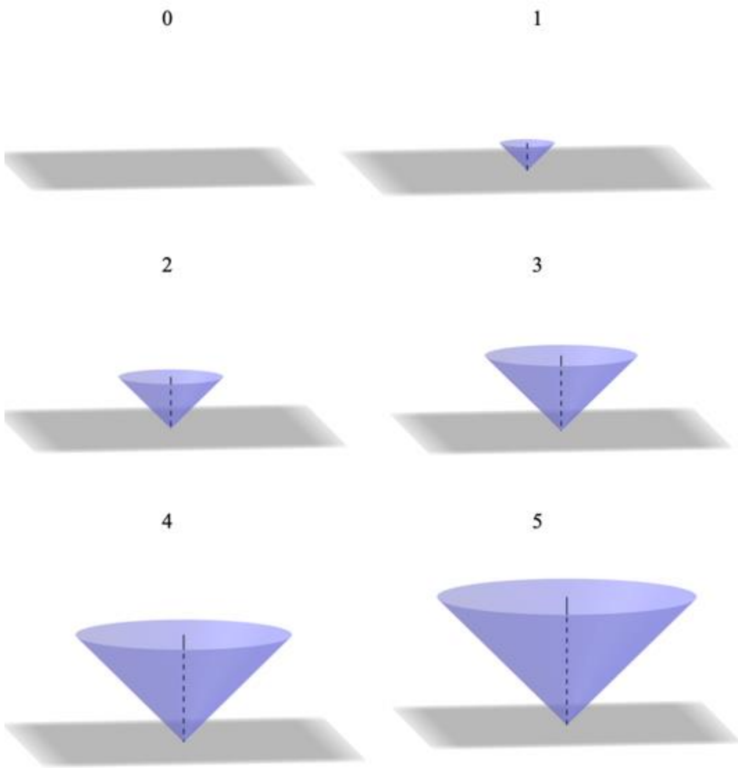
Note: We only present this subset of a larger LIT and HLT related to students' graphing relationships to articulate our design decisions. We refer the reader to Paoletti et al. (2023) for our full LIT and HLTs related to students' quantity construction and how such construction supported their later graphing activity.

We first used a variation of the *Cone Task* described by Stevens et al. (2015) to produce an initial HLT that included specific tasks and tools. We provided middle school students with a Geogebra animation of a growing cone and asked them to consider how the cone's height and surface area covaried (Figure 2). We conjectured students could first construct the cone's changing surface area and height as quantities ($HLT\text{-}1_A$). They could then conceive the cone's height and surface area both increase ($HLT\text{-}1_B$). Finally, we prepared a handout (Figure 4) that included screenshots of the cone at equal integer changes. We conjectured this visual could support students in conceiving that for equal changes in height, the amounts of change of surface area would increase ($HLT\text{-}1_C$).

We emphasize this LIT and HLT represent our initial conjectures about paths for student reasoning and the tools that might support students toward our goals. We describe how we refined both our HLT and our LIT to better support student learning and respond to new ideas we gained in context. As such, this paper describes our iterative design as we produced a theory (i.e., a LIT) and an adaptable task (i.e., an HLT) to support middle schoolers' quantitative and covariational reasoning.

Figure 4

The Cone Task handout where the numbers represented the integer height values of the cone shown in that screenshot.



Methodology and Methods

To refine our LIT and HLT, we engaged in a series of seven teaching cycles.[2] Our inquiry in this paper was guided by the questions: *How did we adapt the design of our LIT and HLT during teaching cycles? What considerations motivated these changes?* We structured our inquiry as an intrinsic case study (Stake, 1995) as we sought to understand the motivations of our own task design process within a given and bounded research context. Below, we provide details about the participants and context of the teaching cycles. We then describe our data collection and analytical processes.

Case Participants and Context

We conducted six small group teaching cycles with sixth- and seventh-grade students (ages 11-13) and one concluding whole-class teaching cycle with eighth-grade students (ages 13-14). To recruit students, we partnered with a middle school in the northeastern United States. The school population included over 75% students of color

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[2] We note that while there are a variety of ways to engage in a teaching cycle, we primarily used the teaching experiment methodology as described by Steffe and Thompson (2000).

and 75% of students entitled to free or reduced-price lunch. For small group teaching cycles, we distributed assent and consent forms to all sixth- and seventh-grade students who were interested and willing to participate. We engaged all students who returned both forms. For the whole class teaching cycle, we engaged students who returned assent and consent forms in an accelerated eighth grade geometry course[3].

The same teacher-researcher (TR; the first author) facilitated all seven teaching cycles. We note that the TR, a former grade 7-12 mathematics teacher[4], was not the students' regular mathematics teacher. Throughout the study, the TR collaborated with a research team, who observed the teaching sessions or reviewed videos between sessions. Therefore, in this case, we also consider the TR and the research team as study participants.

Each teaching cycle involved 4-15 sessions with a broad goal of supporting students' covariational reasoning and graphing meanings. Our focus is on the sessions and task directly aligned with our focal LIT and HLT. The task began as the *Cone Task* (Stevens et al., 2015) but eventually became the *Growing Triangle Task* across the cycles. Although we provide greater detail about the changes to our LIT and HLT in our results, we summarize key differences in Table 2. Further, Table 3 provides the student pseudonyms, iteration of the task, and the number of sessions for each teaching cycle.

Table 2

The dynamic situations, quantities under consideration, and tools available for each iteration.

Task **[HLT Iteration]**	**Quantity 1**	**Quantity 2**	**Tools**
Cone Task [HLT-1]	Height	Surface area	Dynamic applet + handout
Cone Task [HLT-1.1]	Height	Surface area	Dynamic applet + handout + 3D printed manipulatives
Cone Task [HLT-2]	Height	Volume	Dynamic applet + handout + 3D printed manipulatives
Growing Triangle Task [HLT-3 & HLT-4[a]]	Base length	Area	Dynamic applet + paper cutouts or 3D printed manipulatives

[a] HLT-4 involved the same quantities and tool as HLT-3 with an additional goal discussed in the results.

[3] Most students in this school take an algebra course in eighth grade. We use terminology consistent with the school's name for the course.
[4] In the U.S., Grades 7-12 span the ages from 12-18.

Table 3

A summary of the seven teaching cycles

Group	Students	HLT iteration	# of Total Sessions (Sessions on task)
1	Candice & Aliyah	HLT-1	15 (Session 13-14)
2	Zion & Reggie	HLT-1	14 (Session 5-6)
3	Eve, Kyle, & Brian	HLT-1.1	4 (Session 1-2)
4	Grace	HLT-1.1	10 (Session 4-6)
5	Ariana	HLT-2	10 (Session 4-5)
6	Vicente & Lajos	HLT-3	12 (Session 4-6)
7	Neil, Aaron, & Nigel[a]	HLT-4	5 (Sessions 4-5)

[a] For brevity's sake, we focus this report on one group of three students from the whole class

We highlight all seven teaching cycles occurred over a 1.5-year period, allowing the results from one cycle to inform the re-design of the LIT and HLT for the next. We conjectured this design could afford clearer distinction of how particular activities supported student learning, as opposed to adapting sequences mid-cycle with the same class of students (e.g., Gravemeijer, 2004). Further, the transition from small group to whole class teaching cycles allowed us to explore the efficacy of our revised HLT and LIT at greater depth.

Data Collection and Analysis

During each session, the TR provided students with worksheets, blank papers, and tools, including a computer to display digital applets (Table 2). Additionally, for the whole-class teaching cycle, he provided large whiteboards for student collaboration. The TR video- and audio-recorded all groups, capturing utterances, motions, and written-work for analysis.

Our goal across the study was to explore the efficacy of our HLT and LIT by characterizing student reasoning and relating their reasoning to instructional decisions. Therefore, the research team engaged in both on-going and retrospective analysis of the data during and after each cycle.[5] In each case, we documented our characterizations and design decisions, including rationales, in memos. Following this process, the research team subsequently modified the HLT and/or LIT prior to the next cycle.

[5] As our goal is to provide methodological insights, we do not provide detailed descriptions of the student progress, instead referring the reader to our other papers for such descriptions (e.g., Paoletti et al., 2022; Paoletti et al., 2023).

For this report, we re-examined our records of tasks, student thinking, and researcher memos across the study. First, we summarized task changes made between cycles to anchor our analysis (i.e. Table 2). Then, to respond to our first guiding question, we reconstructed and organized changes we made to our HLT and LIT in relation to the task changes. For our second guiding question, we re-examined this dataset alongside the documented motives behind our changes. Consistent with case study methodology, we engaged in combinations of direct interpretation and categorical aggregation (Stake, 1995), looking for patterns within and across our task and theory changes and memo reflections. Ultimately, we observed four considerations in our modifications during the study, which we describe in the results.

To ensure the trustworthiness of our analysis, we maintained detailed and logical chains of evidence connecting our stated considerations to supporting data (Yin, 2018). Further, we expanded our research team during analysis to include a balance of authors who had engaged in the original cycles of design and analysis (Author 3) and authors who had not participated in the data analysis process (Author 2 and Author 4). This triangulation structure encouraged multiple perspectives for our findings and permitted member checking of conclusions (Mok & Clarke, 2015). We discuss our findings across both guiding questions in the following section.

Results

We describe modifications to our LIT and HLT during a design cycle, highlighting how particular considerations influenced our decision-making process across multiple teaching cycles. At a high level, we describe how, after several attempts at redesigning the *Cone Task* from our initial HLT, we reimagined the context entirely by introducing and refining the *Growing Triangle Task*. We detail changes to each task and corresponding HLT (Table 2) in terms of focus quantities and available tools and describe four considerations that influenced these changes. These considerations were supporting student thinking towards the goals of our HLT and LIT (Consideration 1), eliciting students thinking, which is required for characterizing student reasoning (Consideration 2), keeping instruction efficient towards our goals (Consideration 3), and exploring new possibilities (Consideration 4). We describe how each consideration influenced our decision-making process.

The Cone Task

We conducted five teaching cycles using variations of the *Cone Task* (Table 1). Over the course of the study, we modified our initial HLT in several ways. First, we adjusted the task's instructional supports to include 3D manipulatives to better support and elicit students' conceptions of quantities identified in the initial HLT (HLT-1.1; Considerations 1 and 2). Next, we exchanged the focal quantity of surface area for volume to leverage a quantity that students had described more readily when using the manipulatives (HLT-2; Consideration 3). As we reflected on our work with students, we ultimately began to perceive limitations in applying the *Cone Task*. However, each modification also sparked new ideas (Consideration 4) we could carry forward into later iterations of the HLT, and, eventually, our LIT.

Relating Cone's Height and Surface Area (HLT-1)

We conducted four teaching cycles based on HLT-1 (Table 2). Like Stevens et al. (2015), we asked the students to describe quantities when watching the animation (Figure 2) then prompted their attention to the directional and amounts of change between the cone's height and surface area. In the first two teaching cycles (applying HLT-1), we provided students with the digital applet (Figure 2) and worksheet (Figure 4).

Through our work with the first two groups, we wondered if students' descriptions of "surface area" were imagined upon a three-dimensional shape. Candice and Aliyah (Group 1) began by applying the words "surface area" to a quantity we inferred to be the cone's slant height. Although Zion and Reggie's (Group 2) statements indicated they interpreted the situation in three-dimensional space (e.g., describing one quantity as the "circumference of the top of the cone"), they also referred to the cone as a "triangle." Whereas Candice and Aliyah described "surface area" in a way that conflicted with our goals and intentions (Consideration 1), Zion and Reggie's descriptions did not provide the clarity we sought to understand the quantity (or quantities) they were constructing and reasoning about in their activity (Consideration 2).

Given the centrality of surface area to HLT-1_A, the TR conceived a need for additional instructional support in-the-moment to develop students' meanings for surface area. In both teaching cycles, the TR asked students to shade and color-coordinate surface areas and changes in surface area on their handouts (Figure 5). Despite this shading activity, the quantity with which the students were operating was still often ambiguous from our perspective. For instance, when shading the cones (Figure 5), Reggie described "how many [of the] first triangles [*referring to the shaded region in the first image of the cone*] will fit into the bigger triangles" and used this reasoning to describe the "space left" after fitting the smaller triangles into the bigger triangles. In addition to referencing "triangles," Reggie described the "space" as both volume and surface area. Such descriptions created ambiguities as we attempted to characterize the quantity (or quantities) with which students were reasoning (Consideration 2).

Between teaching cycles, the research team also questioned if the shading activity was instructionally efficient (Consideration 3). The shading activity was effortful for each pair, taking between 7 and 9 minutes to complete (nearly one-third of a session). Time spent toward the construction of surface area (HLT-1_A) meant that less time could be devoted toward the other instructional priorities we had outlined in HLT-1.

Figure 5

Zion and Reggie's shading activity

Despite these challenges, we highlight the shading activity supported both groups to satisfy key components of the LIT. Notably, each student described how two quantities covaried with respect to directional (LIT_B) and amounts of change (LIT_C). For example, Zion and Reggie argued the amounts of change in "surface area" increased by more as the height grew, thereby characterizing the relationship the TR had intended. As such, the students' activity satisfied most of the immediate goals of the HLT. Further, by reasoning about the directional and amounts of change of two quantities, each student also achieved the broader goals in our LIT. Given this success, the research team worked at re-designing the task to better support and elicit students' construction of quantities in more efficient ways.

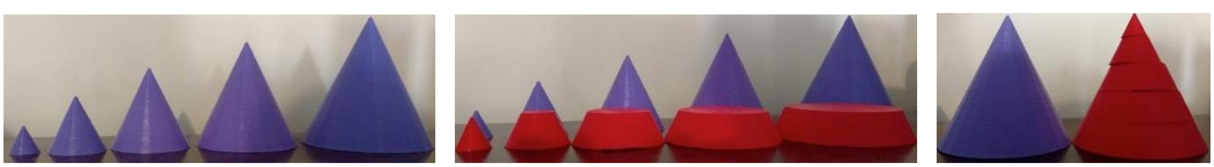
Adding 3D Manipulatives (HLT-1.1)

In HLT-1.1, we maintained the progression of student reasoning but added new tools for students. We conjectured the first two groups could have satisfied the goals of the HLT in a more efficient manner if we had provided more effective tools to support their conceiving of surface area in ways compatible with our intentions. We recognized we asked students to conceive a 3D cone relying upon only 2D images of the cone in digital and physical tools. We anticipated providing students with 3D manipulatives representing the cone could support them to conceive of 2D surface area in a 3D object in line with our HLT (Consideration 1), provide us with clearer evidence of the quantities students were constructing (Consideration 2), and do so efficiently (Consideration 3).

To provide a 3D tool for students, we 3D printed manipulatives of (1) entire cones intended to represent the cone at integer height values (Figure 6, left) and (2) truncations intended to represent the chunks added on to the previous cone to obtain the next cone (Figure 6 center and right). We conjectured the manipulatives could support students in constructing the cone's surface area as a 2D quantity on a 3D object clearly and efficiently (HLT-1_A). We conjectured the different types of pieces might support them in constructing both directional (HLT-1_B) and amounts of change of surface area with respect to height (HLT-1_C). We engaged Eve, Kyle, and Brian (Group 3) and Grace (Group 4) with these new tools in combination with the previous supports (HLT-1.1).

Figure 6

The 3D-printed cone manipulatives



Each group identified the 2D surface area of the 3D cone by pointing and motioning over the manipulatives as they described the cone's outer area as something they could measure. Despite this, we again experienced a complexity that the quantity (or quantities) with which students were reasoning was still either ambiguous from our perspective (Consideration 2) or clearly not the outer surface area as we hoped to support in our HLT (Consideration 1). Group 3, for instance, gestured with their hands to indicate the entire 3D-printed piece when referring to "surface area", not just its outer surface. We could not distinguish if, to these students, "surface area" referred to a quantity such as mass or volume instead. In contrast, Grace compared the circular top faces of consecutive truncations by overlaying one on the other (Figure 7) explaining the amounts of change of “surface area” increased for equal changes in height. Here, Grace conceived of surface area as the area of the truncations wider base, which differed from our intended quantity.

Figure 7

One of the stacks Grace used to compare amounts of change in “surface area.”

We note again both groups leveraged their constructed quantities to reason about directional (HLT-1_B) and amounts of change (HLT-1_C) in ways compatible with our HLT and LIT. However, students’ construction of surface area still presented difficulties for us. We considered adding more instructional supports for students’ construction of surface area, but we grew concerned that additional supports would impact instructional efficiency (Consideration 3). In considering our next steps, we reflected upon how we could modify the learning progression of our HLT to fulfill the central goals of our LIT, addressing our current challenges without creating new ones.

Trading Surface Area for Volume (HLT-2)

In HLT-2, we opted to replace surface area with volume. Below is our revised HLT, with changes italicized:

HLT-2_A – Construct *volume* and height;

HLT-2_B – Conceive that *volume* and height both increase;

HLT-2_C – Conceive that for equal changes in height, the amounts of change of *volume* increase.

Based on Group 3’s activity and the solid nature of the manipulatives, we conjectured volume might be more readily accessible to students. We note if a goal of our LIT included students constructing surface area, we would have 3D-printed shells of cones and of truncations rather than solid pieces to try to better support students’ construction of surface area. However, our LIT was focused on supporting students to reason about directional and amounts of change of two quantities, without specifying the quantity had to be surface area. Hence, we felt the change in quantity could better support and elicit student reasoning (Considerations 1 and 2) while preserving the instructional efficiency the addition of the 3D manipulatives had achieved (Consideration 3).

We engaged Ariana (Group 5) with HLT-2. Ariana spontaneously identified the cone’s “weight” (as opposed to volume) as a quantity, reasoning with weight throughout her activity. Ariana identified the cone’s height and weight were growing as the cone grew aligning with HLT-2_B. She described that because the red truncated cones have “different sizes, their weights have to be different,” and that their “weight is increasing” for consecutive changes in height. Using this reasoning, Ariana identified that as the height of the cone increases by equal amounts, the weight of the cone increases by greater amounts, aligning with HLT-2_C. However, we note due to her focus on weight, rather than volume (HLT-2_A), the HLT was not achieved.

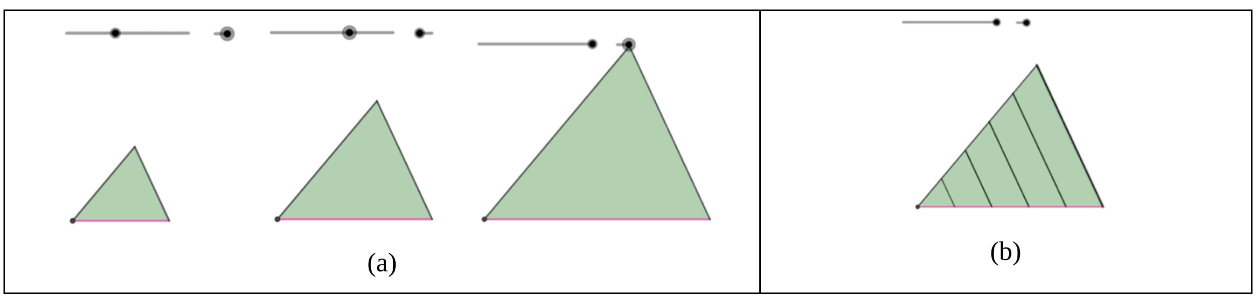
Reflecting on Ariana’s activity, we wondered if the 3D context afforded too many loosely related quantities with which to reason. Collectively, it was difficult for us, and for students, to tease apart different quantities such as surface area, volume, weight, and others. As before, we revisited the goals of our LIT to inform further modifications, which sparked the idea to transition from a 3D to a 2D context (Consideration 4).

The Growing Triangle Task

We engaged two groups in the *Growing Triangle Task*[6]. In this task, students interact with a GeoGebra animation showing a growing triangle (https://www.geogebra.org/classic/pwzzdjaz; Figure 8a). The *Growing Triangle Task* had new focal quantities of base segment length and area but preserved the intentions and structure of our original LIT. We anticipated the shift to a 2D context would allow for students to more easily construct (Consideration 1) and describe situational quantities (Consideration 2) that were familiar to them. Further, we opted to continue to use the same types of instructional supports (i.e., dynamic applet, worksheets, physical manipulatives), as we anticipated these would increase instructional efficiency in supporting students’ quantity construction (Consideration 3).

Figure 8

(a) Three screenshots of the applet’s growing triangle and (b) the applet with ‘trace’ used to highlight amounts of change of area.



Although we initially only modified the task and accompanying HLT (HLT-3), we eventually observed student activity that extended beyond our expectations and inspired new possibilities for the LIT itself (see HLT-4). The *Growing Triangle Task* demonstrated potential to elicit students’ covariational reasoning with respect to directions and amounts of change in supported and increasingly specific ways.

[6] We note that task Ellis et al. (2020) describe a similar exercise within a task they developed independently of this project. We refer the reader to this piece for additional visions of how a “growing triangle” scenario could be applied to reach different learning goals with students.

Changing the Context (HLT-3)

We present our HLT-3, with changes from HLT-2 italicized:

HLT-3_A – Construct changing *base length* and *area* of a dynamic *triangle*;

HLT-3_B – Conceive that *base length* and *area* both increase;

HLT-3_C – Conceive that for equal changes in *base length*, the amounts of change of *area* increase.

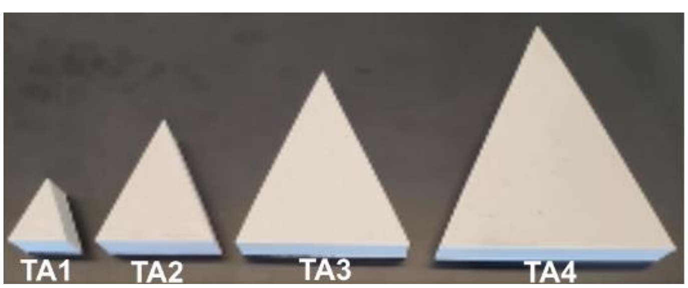
We intended to support students' construction of the triangle's growing area and base length (HLT-3_A), with them identifying both as increasing quantities (HLT-3_B). To support students in observing increasing amounts of change of area (HLT-3_C), we included a second smaller slider enabling students to increase the increment by which the pink length increases (i.e., to equal integer chunks instead of smoothly). Students could trace the triangle's right side to visually identify amounts of change of area (i.e., the size of the consecutive trapezoids in Figure 8b).

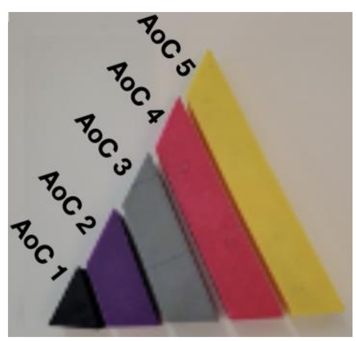
We planned to leverage 3D-printing to use what we had learned about the promise of 3D manipulatives in previous cycles. We printed triangles (Figure 9, left) to represent the growing triangle at four equal increases in base length. Additionally, we printed amounts of change blocks (AoC1-5; one triangle and four trapezoids, Figure 9, right), which we intended to represent the amount added to each gray triangle to get to the next consecutive triangle.

We applied HLT-3 with Vicente and Lajos (Group 6).[7] Their activity affirmed the affordances of the 2D context that we conjectured. First, they described several quantities they conceived, including "length," "width," "height," "perimeter," and "area." The students elaborated area, using the smallest triangle (TA1 in Figure 9a) as a unit to measure the area of the other pieces (HLT-3_A). This activity indicated the students conceived of area in ways compatible with our intention.

Figure 9

The 3D-printed manipulatives for the Growing Triangle Task with added labels for reference.





[7] It was our intention to use the 3D manipulatives at the outset. However, due to issues with the timing of the printers and the teaching cycle's schedule, we did not have the 3D manipulatives and instead used paper cut-outs analogous to the pieces in Figure 9 during the first session.

The students next conceived the directional and amounts of change of area and base length. Vicente noted as the animation played, "[the area] keeps going up, you get like bigger and bigger." Additionally, they noted that the pink base length increased with the area (HLT-3_B). The TR then overlaid the first two paper triangles, as seen in Figure 10a, and asked, "How much has the area changed by?... Just color in the, how much the area changed by from the first to the second." Responding to this Vicente shaded the area seen in Figure 10a. After this, the TR placed the two cutouts on top of the third triangle (paper version of TA3, Figure 10b) and asked Lajos to represent the next amount of change. Lajos shaded this area (Figure 10b). Each student identified that the amount of change of areas were "getting bigger" as the triangle grew, thereby identifying the increasing amount of change of area (HLT-3_C).

Figure 10

The students' shading the (a) first and (b) second amount of change using the paper cut-out triangle manipulatives.

(a) (b)

When analyzing each student's activity, we concluded their reasoning was compatible with all aspects of our LIT and HLT. The new context provided quantities more readily constructable for students (Consideration 1) and afforded us more clarity regarding which quantities students were constructing (Consideration 2). The new context and supports also maintained instructional efficiency in accomplishing the goals of our HLT (Consideration 3).

With our primary goals accomplished, we began to reflect on an interesting aspect of Vicente's and Lajos's activity. Specifically, when they used the small triangle to measure the amounts of change pieces, they generated numeric measurements that could enable more precise comparisons. This activity sparked a new idea for us (Consideration 4), as we conjectured the potential to concretely measure area could afford opportunities for students to identify constant second differences, the defining characteristic of quadratic change (e.g., Ellis, 2011; Fonger et al., 2020). We revised our HLT and LIT and explored this potential.

Exploring New Possibilities (HLT-4 and LIT Revised)

We designed HLT-4 to extend our progression to explore the potential for students to conceive of constant amounts of change of amounts of change. Below is our revised HLT, with changes italicized:

HLT-4_A – Construct changing area and base length of a dynamic triangle;

HLT-4_B – Conceive that base length and area both increase;

HLT-4_C – Conceive that for equal changes in base length, the amounts of change of area increase.

HLT-4$_D$ – Construct constant amounts of change of the amounts of change of area for equal changes in base length.

We applied HLT-4 during a whole-class teaching cycle. Because of the clear, supported, and efficient reasoning we observed with Group 6, we used the same tools (i.e., a digital applet and 3D manipulatives). Here, we describe one group's activity—Neil, Aaron, and Nigel (Group 7).

Each student provided evidence they conceived of area as a quantity compatible with our intentions. Further, each student also described directional and amounts of change. First, like Group 6, they used the smallest manipulative piece as a unit to measure both the triangle's area and the amounts of change of area through tiling actions (HLT-4$_A$). Next, they identified both the triangle's area and base length were increasing (HLT-4$_B$). Finally, when prompted to describe the amounts of change of area for equal changes in base length, Neil described, "It gets larger… even larger," reflecting Neil's conceiving an increasing amount of change of area as the triangle grew (HLT-4$_C$).

To support the new element of the HLT, the TR asked the students to consider, "How do the amounts of change compare to one another?" Responding to this, Neil and Nigel began to stack the trapezoidal manipulatives in consecutive amounts (shown in Figure 11). The following conversation ensued:

Neil: [*Places pink trapezoid on top of yellow trapezoid*] Ooh! Look, look, it's a quadrilateral [*pointing to the piece at the end of yellow trapezoid not covered by the pink trapezoid, Aaron agrees*]. Right, look, if we do it again. [*Nigel places gray trapezoid on top of pink trapezoid*] Same size [*pointing to quadrilaterals created by the difference in the yellow and pink trapezoids and pink and gray trapezoids indicating these areas are the same*]. Put that one on [*Nigel places the purple trapezoid on top of the stack*] Same size [*referring the quadrilaterals formed by the difference trapezoids highlighted in Figure 11b*].

TR: So what did you notice?

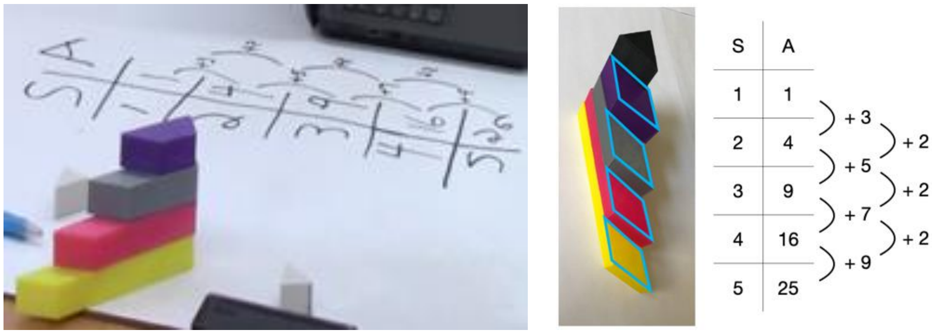
Neil: It's like, this quadrilateral [*pointing to quadrilaterals highlighted in Figure 11b*] keeps going I guess, it's added on to that [*pointing to each layer of the stack of trapezoids*].

TR: That, that piece we're adding on to the amounts of change is always the same? [*Nigel nods in agreement as TR speaks*]

Neil: Yeah.

Figure 11

(a) The stacked manipulatives and table, and (b) a recreation of an overhead view of (a), with equal amounts of change of amounts of change highlighted in the manipulatives.



(a) (b)

Neil described the amount being added to each trapezoidal manipulative as equal (i.e., the amount of change of the amount of change of area was constant; HLT-4_D). Both Nigel and Aaron expressed agreement with Neil's conclusion at different points, as well. Later, the students identified these constant amounts of change of amounts of change of area as constant second differences in a table representing the growing triangle's area and base length (Figure 11). This activity likely supported the students in developing meanings for quadratic relationships that entailed constant second differences.

Given the productivity of the new feature in our HLT, we created a revised LIT. We now conjectured our LIT could not only support students to reason about two changing quantities to conceive a relationship entailing both directional and amounts of change, but also to leverage these amounts of change to further characterize specific types of covariational relationships. Below, we added a final component to our original LIT (LIT_D), with changes italicized:

LIT_A – Construct two changing quantities in a dynamic situation;

LIT_B – Conceive a directional relationship between the two quantities constructed in LIT_A;

LIT_C – Conceive of amounts of change of one quantity with respect to equal changes in the second quantity from LIT_B;

LIT_D -*Conceive of amounts of change of amounts of change of the quantity in* LIT_C *and characterize how the amounts of change of amounts of change are changing.*[8]

Discussions, Implications, and Areas for Future Research

Our goal in this report was to describe our process of designing and refining tasks and instructional progressions through design and teaching cycles to support students' covariational reasoning. Addressing the first guiding question regarding how we adapted the design of our HLT and LIT, we detailed how we transformed the *Cone Task* (Stevens et al., 2015) into the *Growing Triangle Task*, refining the task scenario, tools, and underlying instructional

[8] We note this approach is similar to Ellis (2011)'s description of students reasoning about the difference in the rate of growth to conceive of quadratic relationships.

progressions to create evidence-based instructional products. We applied theories of quantitative and covariational reasoning to ground our initial designs and goals (e.g., Smith & Thompson, 2008; Thompson & Carlson, 2017), which centered around coordinating amounts and directions of change with clear quantities in mind (Carlson et al., 2002). This supported us to develop not only several iterations of an HLT to promote student learning but also to ultimately revise the generalized progression for learning (LIT) we envisioned as underlying these tasks.

Our revision process did not occur arbitrarily; rather, we found that we engaged in design decisions in accordance with several considerations that guided our thinking. Addressing the second question, we highlighted our attention to four primary considerations in each new iteration and refinement: supporting student thinking towards the goals of our HLT and LIT (Consideration 1), eliciting students' thinking (Consideration 2), keeping instruction efficient towards our goals (Consideration 3), and exploring new possibilities (Consideration 4). We note that the considerations we identified in our case also align with the structure of mathematics instruction design cycles described in the surrounding literature. That is, LITs and HLTs can serve as reciprocal foundational tools to organize goals, tasks, and progressions of student learning (Gravemeijer, 2004; Nickerson & Whitacre, 2010).

In the paragraphs that follow, we provide implications for researchers, teachers, and curriculum designers. We first highlight the importance of attending to the quantities students are constructing. We then provide implications regarding the use 3D printed manipulatives in our study, highlighting the affordances they had in both 3D and 2D situations. We conclude with implications for the design and implementation of tasks in DBR cycles.

Prior research has shown that students often construct quantities in ways inconsistent with researcher's, teacher's, or curriculum designer's intentions (Moore & Carlson, 2012; Paoletti, 2015; Paoletti et al., 2024). Our results further highlight the importance for researchers and teachers to carefully attend to the quantities with which students are reasoning. We presented multiple examples in which students described an accurate covariational relationship with respect to our intentions but were clearly not reasoning about the quantities we intended (i.e., Groups 2, 4, and 5). If we were not attending to the quantities student's constructed, we may have taken their statements about the final relationship as reflecting an understanding of the situation compatible with our intentions. However, by carefully attending to the quantities students constructed, we realized the situations we provided often allowed for students to reason about surrogate quantities that were related to our intended quantity without constructing it. Hence, in this report we show how revising tasks with attention to students' quantity construction can support goals related to students' covariational reasoning.

Our results also highlight the importance of teachers, researchers, and curriculum designers carefully considering the tools they provide students when intending to support their constructing and reasoning about quantities. In our study, when using 2D representations of 3D quantities, we needed to provide students ample time to construct and make sense of the 3D quantities through the 2D images. The use of 3D printed manipulatives proved effective and time-efficient in supporting students' construction of quantities in the situation. Although we eventually abandoned the

3D situation, we conjecture researchers interested in supporting students' conceptions of surface area could use 3D-printed shells of cones as a potential tool; we call for research exploring this possibility.

In addition to using 3D-printed manipulatives to represent a 3D scenario, we also used 3D-printed manipulatives in the *Growing Triangle Task*. An unexpected affordance of these manipulatives was the ways students could use them to visualize the constant amounts of change of amounts of change of area (Figure 11b) that is the defining characteristic of quadratic growth. Such a task could be used in conjunction with tasks described by others (e.g., Ellis, 2011; Fonger et al., 2020) to support students' in developing meanings for quadratic growth. We call for additional research further exploring this possibility, including exploring the efficacy of our final LIT.

Our continued use of 3D manipulatives even in a 2D situation is just one example from the above story in which we were able to glean valuable information for a future design cycle despite our not deeming that cycle as successfully achieving our goal. By presenting our design decisions in detail, we hope to provide researchers, teachers, and curriculum designers with ways to reflect on why a particular teaching cycle was not successful in achieving the HLT and also to generate ideas to revise the HLT (or LIT) for the next cycle. By carefully considering affordances and constraints of particular contexts and tools, we were able to design a task that not only supported our goals but enabled us to explore possibilities that exceeded our initial intentions.

Concluding Remarks

We presented our findings as a case study of our own design cycle with the goal of providing an account that could examine the process with reflexivity and transparency (e.g., Tracy, 2010). We note, as Stake (1995) describes, case studies are designed to understand the inner workings of particular processes and phenomena. As such, future works could continue to describe considerations other researchers examining and supporting student learning use during design cycles across broader projects, groups of researchers, or even mathematical content foci. Although our work is situated within a particular theoretical perspective within mathematics education (students' quantitative and covariational reasoning), we advance this piece as a starting point in contributing to the need for increased reporting of design cycles in mathematics education more generally, including strong connections between theories and tasks (e.g., Kieran et al., 2015).

As we have shown through this project, there is great potential in the iterative structure of DBR research projects intended to support student learning to construct instructional products with focus, applicability, and relevance through an ongoing process of refinement. Structures such as LITs and HLTs can specify and articulate the relationships between theorized progressions, tasks, and their enactment. Moreover, we contend that reporting of these structures and the process behind them can support new directions in how we not only discuss but also conduct DBR work in our research.

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