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Grade 12 Students' Misconceptions when Modelling their Calculus Ideas into their Learning of an Optimization: A Real-Life Problem

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Abstract: Optimisation is a key topic in calculus and an essential component of the Grade 12 mathematics curriculum in South Africa. However, many students struggle with the modelling of the calculus concepts correctly when solving real-world optimisation problems. The reason behind this is not far from some misconceptions exhibit by some students when calculus. It is on this note that this study investigates the misconceptions that Grade 12 students exhibit when applying calculus to optimisation problems. A qualitative research approach was used, involving a test and follow-up interviews with five students selected from a class of 35 based on their responses. Data collected and analysed by the researchers revealed that students commonly confuse the derivative with the function itself, struggle with interpreting the concept of rate of change, and misapply algebraic techniques when solving optimisation problems. These misconceptions were therefore classified into conceptual errors, generalisation errors, language errors, and other hidden errors. The findings suggest that these misconceptions stem from a combination of inadequate conceptual understanding, procedural reliance, and language barriers. The study recommends and targets the instructional interventions focusing on deep conceptual understanding, explicit connections between calculus and real-world applications, and the refinement of mathematical language use to improve students' modelling abilities in optimisation.

Keywords: Calculus, misconception, Grade 12, modelling, optimisation
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Introduction

Recent transformations around the world had led the educational authorities to see the teaching and learning of calculus as very important to them, as it plays a crucial role in preparing students for higher-level mathematical applications (Cline et al., 2020; Rasmussen et al., 2014; Motseki & Luneta, 2024). In South Africa, calculus has been integrated into the Grade 12 mathematics curriculum as part of the Curriculum Assessment Policy Statement (CAPS), emphasizing its significance in developing students' problem-solving and mathematical modelling skills (DBE, 2020). Given its real-world applications, many countries have also introduced calculus at the high school level to enhance students' readiness for undergraduate studies and careers in science, technology, engineering, and mathematics (STEM).

Despite the importance of calculus, students often struggle with key concepts, particularly in optimisation, leading to errors in mathematical modelling. As shown empirically below, a variety of high schools and institutions that teach calculus, as a core subject in mathematics, have reported varying rates of academic accomplishment. According to reports from some researchers, it was confirmed that the pass rate of students in calculus varies from one institution to the others, but typically it ranges from 50% to 75%. On this ground, it was reported that the retention rates of students are below 40%, which had led to a statistical record of 40% to 50% of students failing calculus classes in their first year (Hagman et al., 2017). In view of this, it makes it more difficult for students to progress in STEM professions. It is regrettable to state that most of the failure rates observed were caused by basic mistakes and misunderstandings that students make when learning calculus. Similarly, further research suggests that these difficulties encountered by students in some mathematical concepts has been generated from various misconceptions, and these include language-related errors, misapplications of concepts, and challenges in multi-step problem-solving (Cline et al., 2020; Mkhatshwa, 2016; Motseki & Lunata, 2024). In South Africa, Motseki & Lunata (2024) argue that students' struggles with optimisation are influenced by the abstract nature of the topic and their cognitive development. Additionally, the Department of Basic Education recognises the need for strong emphasis on calculus, since the DBE dedicating a significant portion to it (up to 40%) in Grade 12 mathematics curriculum to its teaching (DBE, 2020).

Given to these challenges, this study seeks to examine the common misconceptions Grade 12 students have when learning optimisation in calculus and their impact on mathematical modelling. By understanding the nature of these misconceptions, the study aims to provide insights into how teaching strategies can be improved to enhance students' comprehension of optimisation. The research is guided by the following question:

- What are the forms of misconceptions among Grade 12 students when modelling their calculus ideas into their learning of optimisation?

To address this question, a qualitative research approach was adopted, gathering insights from Grade 12 students in a South African high school. The findings of this study aim to contribute to the ongoing discourse on improving calculus instruction and supporting students in developing a deeper understanding of mathematical modelling in real-life contexts.

The Views of the Literature on the Teaching and Learning of Calculus as a Course

Although the teaching of calculus in some developed countries like the United States, Uruguay, France, Germany, and Singapore has been taken seriously among the teachers and educational authorities, the fact remains that some countries still dodge some important aspects of calculus due to its difficult nature among the students (Bressoud et al., 2016; Thompson & Harel, 2021). Therefore, we have taken it upon ourselves to report the state of teaching optimisation among the South Africa Grade 12 students in South Africa. Some countries believe that calculus as a major topic in mathematics education should be given a form of transitional privilege by allowing its teaching process to reflect in both secondary and tertiary educational curricula, with a special focus on the difficult aspects (Thompson & Harel, 2021). Therefore, it is important to note that South Africa as a country believes that complex mathematics like calculus should be given to mature students for the purpose of solving complex mathematics problems through an intensive teaching and learning process (Rasmussen et al., 2014). It is on this basis that some researchers and educators have described the process of teaching and learning calculus in South Africa, as reported by Makonye and Luneta (2014).

Furthermore, in the Republic of South Africa, the teaching and learning of calculus and optimisation have come with some exceptions. For instance, in the Grade 12 curriculum, topics related to calculus were added to CAPS, with some mathematics aspects concentrating on the teaching of some aspects in a realistic way. Further, it could be said that the teaching of calculus was found to be vital due to its link with some aspects of engineering, accounting, and mathematics-related courses (DBE, 2015; Makonye & Luneta, 2014). But it is unfortunate to report that some Grade 12 students struggled with several areas of optimisation due to a lack of expertise in some newly learned calculus concepts; as a result, they were unable to apply their knowledge to the problem at hand in calculus (Ndlovu & Brijlall, 2016). On that note, little research in South Africa has investigated different areas of teaching and learning calculus, such as the usage of symbols to analyse how symbols are employed in mathematics, such as algebraic and trigonometric expressions (Ndlovu & Brijlall, 2016). However, teaching and studying calculus at the high school level remains hard and difficult to understand (Smith, 1994). As a result, the only way to improve the teaching and learning of the concepts of calculus is by concentrating more on the modern approach and intensive instruction approach, and by increasing research on the teaching and learning of calculus to resolve the misconceptions. Hence, this study focuses on identifying forms of misconceptions among Grade 12 students when modelling their calculus ideas into their learning of optimisation.

Conceptualisation of Optimisation as a Grade 12 Topic in the South African

Optimisation is a significant mathematical concept that is included in the Grade 12 curriculum in South Africa, and part of its goal involves finding the best solution from a set of feasible solutions, when determining the maximum and minimum values of functions. The concept of optimisation is not only crucial for academic purposes but also for real-world applications where resources are limited, and efficiency is key. It is on this ground that the South African curriculum, centres its focus on high knowledge and high skills, and ensures that learners are equipped with the ability to solve complex problems on optimization (Learning Channel, 2010). Additionally, resources like the DBE curriculum tool reports that the mathematics self-study guides will provide comprehensive material for both teachers and learners to deepen their understanding on optimisation and other mathematical concepts (DBE, 2020). This aligns with the curriculum's goal of preparing students for higher education and the professional world, where such skills are highly valued around the world.

Calculus, particularly the field of differential calculus, is fundamental to optimisation problems, because it helps to identify the maximum or minimum points of a function, which are essential in determining the most efficient solutions to real-world problems (Strang, 2020). For instance, businesses may use calculus to maximise profit or minimise costs. This is achieved by first defining the function that needs optimisation, which could represent anything from cost to revenue and some materials usage. Then, the derivative of this function is taken to find the critical points where the slope is zero or undefined, indicating potential maxima or minima. By further analysing these points with tests like the second derivative test, one can determine whether they represent a maximum or minimum value. This process is crucial in making decisions that require the most effective use of resources, and it exemplifies the practical application of mathematical concepts learned in academic settings.

An example of optimisation in the Grade 12 Mathematics curriculum of South Africa could involve a problem where students were asked to determine the dimensions of a box with the largest possible volume under certain constraints, such as a limited amount of material available for construction (Learning Channel, 2010). This problem would require the application of calculus, specifically the use of derivatives to find the maximum value of a function that models the volume of the box. Students would learn to set up an equation for the volume, differentiate it to find the critical points, and then test these points to find the maximum volume possible (Strang, 2020). This type of problem not only enhances their mathematical skills but also teaches them how to apply these concepts to real-world situations, such as product packaging or construction projects, where optimisation is key to efficiency and cost-effectiveness.

At a Grade 12 level, the topic of optimisation entails some important aspects, like maximum and minimum aspects, and this also includes some aspects of algebra and quadratic function. And this was done to increase the students' understanding of some practical areas of derivative, rate of change, and graphical aspect which show the maximum and minimum function of a graph. Practically, optimisation in the South African school curriculum report includes problems on how to calculate simple equations leading to quadratic problems, which check the dimensions of an object, and problems leading to measurement of the volume of water and liquid (Learning Channel, 2010). All these comprise different formulae which combine to form calculus topics in the Grade 12 syllabus in South Africa.

Going by the arguments stated above, one could argue that optimisation remains vital when teaching calculus in the South Africa school curriculum. But the fact remains that when answering some major real-world problem in optimisation, some aspects of precalculus and quadratic functions come into play. It is on this ground that the Department of Education in South Africa introduced formular like quadratic formular and equation to the topic of optimisation to facilitate students' learning process.

Addressing Misconceptions in Calculus and Optimisation

The teaching and learning of calculus have been extensively explored in mathematics education research, with scholars identifying both its significance and the persistent difficulties students face when engaging with optimisation problems (Bressoud et al., 2016; Thompson & Harel, 2021). Calculus is regarded as foundational for higher-level mathematics and its applications in science, engineering, and economics. However, despite its importance, students continue to struggle with conceptual understanding, leading to widespread misconceptions (Cline, Zullo, & Huckaby, 2020). This section critically synthesizes existing studies on calculus education, highlighting contradictions, limitations, and the need for further research as reported below.

The Role of Calculus in Secondary Education and Higher Learning Preparedness

Calculus is a significant component of secondary school curricula worldwide, particularly in South Africa, where it is allocated a considerable portion of the Curriculum Assessment Policy Statements (CAPS) (DBE, 2020). The Department of Basic Education (DBE) has explicitly stated that calculus-related topics account for approximately 40% of the mathematics curriculum for Grade 12 students, emphasizing their importance in national assessments and university readiness (DBE, 2020). This focus is intended to strengthen students' mathematical reasoning and problem-solving skills, which are essential for success in higher education and STEM-related careers. However, the effectiveness of this emphasis remains debatable because some studies suggest that a strong calculus foundation enhances student preparedness for tertiary education Brijlall & Ndlovu (2013), while others argue that many students develop only procedural knowledge rather than deep conceptual understanding (Tall & Katz, 2014). For example, research by Makonye and Luneta (2014) found that South African students struggle with applying calculus concepts in real-world contexts, indicating that the existing curriculum may not sufficiently support conceptual mastery. This contradiction suggests that while calculus is given significant weight in the curriculum, its practical and conceptual integration remains a challenge in South Africa.

Common Misconceptions and Learning Challenges in Calculus

A substantial body of research has identified the nature and causes of student misconceptions in calculus, particularly in optimisation. Olivier (1989) defined misconceptions as persistent incorrect beliefs that hinder learning, and this concept has been widely applied to mathematical education (Brodie, 2009). Students often develop misconceptions due to ineffective instructional approaches, overgeneralization of mathematical rules, and difficulties in conceptualizing abstract ideas (Cline et al., 2020; Jameson, Machaba, & Matabane, 2023).

One of the most prevalent misconceptions in calculus involves the interpretation of derivatives. Several studies have shown that students misinterpret the derivative as a function rather than a measure of rate of change (Mkhatshwa, 2016; Thompson & Harel, 2021). This issue becomes particularly pronounced in optimisation problems, where students struggle to distinguish between a function and its derivative, leading to incorrect applications of calculus principles. Some researchers argue that these errors arise because students rely on rote memorization rather than a deep understanding of mathematical relationships (Herheim, 2023).

Another critical issue is the misapplication of algebraic techniques when solving optimisation problems. Makonye and Luneta (2014) observed that students frequently fail to translate word problems into correct mathematical models, leading to errors in differentiation and the solving of the equation involved. This aligns with findings by Bezuidenhout (2001), who noted that some students often apply algebraic rules incorrectly when working with calculus concepts, resulting in systematic errors in problem-solving. However, some researchers, such as Ndlovu and Brijlall (2016), suggest that these misconceptions stem from weaknesses in pre-calculus instruction, rather than calculus itself. This highlights a key limitation in existing research—whether misconceptions arise from deficiencies in calculus teaching or from foundational gaps in algebra and functions.

The Impact of Language and Instructional Methods on Learning Calculus

Language and instructional approaches play a crucial role in shaping students’ understanding of calculus concepts. Several studies have found that students whose first language is not the language of instruction often struggle with calculus terminology, leading to conceptual misunderstandings (Alt et al., 2014). In the South African context, where many students learn mathematics in English as a second or third language, these linguistic barriers contribute to misinterpretations of mathematical concepts (Makonye & Luneta, 2014).

In addition to language difficulties, the traditional teaching methods used in many classrooms may reinforce misconceptions. Research by Fumador and Agyei (2018) suggests that teacher-centred instruction, which focuses heavily on procedural fluency, often fails to promote a deep conceptual understanding of calculus. In contrast, inquiry-based and problem-solving approaches have been shown to improve student comprehension and reduce misconceptions (Herheim, 2023). This suggests that reforming instructional practices—by integrating real-world applications and emphasizing conceptual reasoning—could be an effective strategy for addressing calculus-related misconceptions.

The Need for Further Research and Pedagogical Improvements

Despite extensive research on calculus education, gaps remain in understanding how to effectively mitigate misconceptions in optimisation. Many studies focus on diagnosing errors but offer limited empirical evidence on intervention strategies. Future research should explore the effectiveness of targeted pedagogical approaches, such as concept-based learning, visual representations, and the use of technology in calculus instruction.
Additionally, there is a need for more empirical data on how calculus proficiency correlates with university success rates in South Africa. While international studies have demonstrated a link between strong calculus skills and STEM achievement (Strang, 2020), local studies on this topic are scarce. Gathering statistical evidence on student performance in calculus and its impact on higher education readiness could help inform curriculum reforms and instructional strategies.

The literature highlights the critical role of calculus in secondary education and the widespread challenges that are faced by some students when mastering optimisation. A few research has identified common misconceptions and their possible causes, contradictions remain regarding the root of these difficulties—whether they stem from deficiencies in foundational mathematics, language barriers, or instructional methods. Addressing these issues requires a shift from procedural teaching to conceptually driven instruction, with an emphasis on real-world applications and problem-solving skills. Furthermore, future research should incorporate empirical data on student outcomes to better inform calculus education policies and practices.

Methods

Research Design

This study employed a qualitative research design, utilizing a case study approach to explore Grade 12 students' misconceptions when modelling calculus concepts into their learning of optimisation. A qualitative case study was chosen because it allows for an in-depth investigation of students' thought processes, misconceptions, and reasoning patterns in a real-life learning environment (Creswell & Poth, 2018).

The study was conducted at a secondary school in Limpopo province, South Africa. The target population comprised 35 Grade 12 mathematics students who were actively studying calculus as part of their curriculum. A purposive sampling strategy was employed to select participants, as this approach allows researchers to focus on individuals who can provide rich, relevant data based on their engagement with the topic (Bernard, 2017; Creswell & Poth, 2018). A diagnostic test on differential calculus and optimisation was administered to all 35 students. This test served two purposes: Identification of common misconceptions and errors: The test responses were analysed to identify typical errors and misunderstandings in optimisation.

Selection of Participants for In-Depth Analysis: Based on the test results, five students were selected for follow-up interviews. The selection criteria were based on a range of performance indicators, not just correctness of answers, but also the types of errors committed, reasoning strategies employed, and evidence of conceptual misunderstandings (Pandey & Pandey, 2015). These students were not necessarily the "best learners" in terms of academic performance but were chosen because their responses exemplified key misconceptions relevant to the study's objectives. The use of the same test for both selection and data collection were justified by its dual function: it provided insight into common misconceptions at a class-wide level, and it enabled a detailed exploration of students' reasoning in the subsequent interviews. However, to strengthen validity, interviews were used to triangulate the findings, ensuring that students' misconceptions were not merely artifacts of test-taking but reflected deeper conceptual misunderstandings.

Data Collection

The diagnostic test consisted of a single real-world optimisation problem, structured to assess students' ability to model a calculus concept into a mathematical context. The use of one test question may be seen as a limitation, but it was deliberately selected because it encompassed multiple dimensions of optimisation: understanding derivatives, interpreting maximum and minimum points, and applying rates of change in a real-life scenario. This approach aligns with prior research indicating that open-ended, context-based tasks are effective in revealing student misconceptions (Mkhatshwa, 2016; Thompson & Harel, 2021).

Following the test, semi-structured interviews were conducted with the five selected students to gain deeper insights into their understanding of calculus concepts, particularly in the context of optimisation (Creswell & Poth, 2018). These interviews were carefully designed to explore the students' reasoning, clarify misconceptions, and examine their conceptual grasp beyond procedural fluency. By engaging in open-ended discussions, the researchers aimed to uncover the cognitive processes that influenced students' problem-solving approaches and to identify specific areas where misunderstandings persisted.

A key objective of the interviews was to probe students' reasoning and justifications for their test responses. Students were asked to elaborate on their thought processes when answering specific questions from the test, allowing the researchers to assess whether their solutions were derived from conceptual understanding or merely from memorised procedures (Davidavičienė, 2018). This approach helped in distinguishing between students who could apply calculus principles flexibly and those who relied solely on algorithmic techniques without a deeper comprehension of the underlying concepts.

Additionally, the interviews were structured to clarify misconceptions and uncover underlying thought processes that may not have been immediately evident in the written test responses. By encouraging students to verbalise their reasoning, the researchers could identify patterns of errors and determine whether misconceptions stemmed from fundamental misunderstandings of calculus principles, incorrect application of rules, or language-related difficulties. This process provided valuable insights into the root causes of errors, which could then inform targeted instructional interventions.

Beyond identifying errors, the interviews aimed to explore students' conceptual understanding beyond procedural fluency. Students were encouraged to explain key concepts in their own words, relate mathematical ideas to real-world applications, and discuss alternative methods of solving optimisation problems. This aspect of the interview process was crucial in determining whether students had developed a flexible and integrated understanding of calculus or if their knowledge remained fragmented and context dependent.

Each interview lasted approximately 30–40 minutes, providing ample time for in-depth discussions. During these sessions, students were not only asked to explain their test responses but also to reflect on their problem-solving strategies. This reflective component allowed researchers to assess students' metacognitive awareness—whether they could evaluate their own approaches, recognise errors, and adjust their thinking (Creswell & Poth, 2018).
To ensure accuracy and facilitate detailed analysis, all interviews were audio-recorded and subsequently transcribed. The transcripts were then examined to identify recurring themes, misconceptions, and variations in students' conceptual understanding. This qualitative data complemented the quantitative findings from the test, offering a more comprehensive picture of the challenges students faced when learning and applying optimisation concepts in calculus.

Overall, the interviews provided critical insights into the cognitive and metacognitive challenges that Grade 12 students encountered in their learning of optimisation. The findings underscored the need for instructional approaches that prioritise conceptual understanding, promote reflective thinking, and address common misconceptions through targeted interventions.

The collected data were analysed using a thematic analysis approach (Braun & Clarke, 2006), which involved several systematic steps. First, the process began with familiarization with data, where test responses and interview transcripts were reviewed multiple times to identify recurring patterns and gain an in-depth understanding of students' reasoning. This was followed by the coding stage, in which errors, misconceptions, and reasoning patterns were systematically identified and labelled. Once coding was complete, theme development was undertaken, categorizing misconceptions into four key types: conceptual errors, generalisation errors, language errors, and other hidden errors. Finally, in the interpretation phase, the identified misconceptions were compared to existing literature on calculus learning difficulties, allowing for a deeper contextual analysis of the findings.

Additionally, APOS theory (Action-Process-Object-Schema) was used as a conceptual framework to interpret students’ mental constructions of optimisation concepts (Arnon et al., 2014). This theoretical lens helped in categorizing students’ misconceptions based on their cognitive processing levels, offering insights into whether their difficulties arose from procedural memorization, an inability to conceptualize mathematical relationships, or struggles with abstract reasoning. The application of APOS theory provided a structured approach to understanding the different ways students process and apply mathematical knowledge in optimisation (Bintoro & Sukestiyarno, 2021).

Validity and Trustworthiness

To ensure validity and trustworthiness, several strategies were employed throughout the study. Triangulation was used by combining test responses and interview data, ensuring that conclusions drawn from the findings were supported by multiple sources of evidence. Additionally, member checking was conducted, wherein students were asked to review and confirm the accuracy of their interview transcriptions, thereby minimizing misinterpretations (Pandey & Pandey, 2015). To further enhance reliability, peer debriefing was undertaken, where fellow researchers reviewed the coding process and thematic development to ensure consistency and objectivity in data analysis. These measures helped to reinforce the credibility of the study and ensure that the findings accurately reflected students' experiences and misconceptions.

Ethical Considerations

Ethical approval for the study was obtained from both the school administration and the relevant research ethics board. Before participation, informed consent was secured from both students and their guardians, ensuring that all participants fully understood the purpose and nature of the study. Confidentiality was strictly maintained by assigning pseudonyms to all participants, protecting their identities throughout the research process (Creswell & Poth, 2018; Pandey & Pandey, 2015). Furthermore, students were granted the right to withdraw from the study at any stage without penalty, emphasizing their autonomy and ensuring ethical integrity in the research process.

The methods employed in this study were carefully designed to ensure a rigorous and ethical exploration of student misconceptions in calculus. By integrating a qualitative case study design, purposive sampling, diagnostic testing, and in-depth interviews, the study provides a comprehensive understanding of how Grade 12 students conceptualize and struggle with optimisation. This multi-faceted approach not only identifies key areas of difficulty but also offers valuable insights for improving calculus instruction and addressing common learning barriers in mathematical problem-solving.

Theoretical Framework: Piaget's Constructivism Theory

This study was guided by the Piaget's constructivism theory, which provides a crucial framework for understanding how students construct mathematical knowledge, particularly in calculus and optimisation. According to Piaget, cognitive development occurs through the processes of assimilation and accommodation, which are essential for learning complex mathematical concepts (Waite-Stupiansky, 2022).

Assimilation refers to the process by which learners incorporate new mathematical knowledge into their existing cognitive structures without fundamentally altering them. In the context of learning optimisation in calculus, students may attempt to apply previously learned algebraic and graphical techniques to solve optimisation problems (Arnon et al., 2014). However, if they merely assimilate these new concepts without adjusting their existing knowledge framework, they may struggle with recognising the nuances of differentiation and critical points in functions.

Accommodation, on the other hand, occurs when students modify their existing cognitive structures to incorporate new information that does not fit within their prior understanding. In calculus, accommodation is necessary when students must grasp the concept of derivatives as rates of change rather than merely procedural calculations (Arnon et al., 2014; Jameson et al., 2023). For example, a student who previously understood maximum and minimum values only in the context of quadratic functions must restructure their understanding when introduced to broader applications of differentiation in optimisation.

The application of Piaget's theory to this study suggests that misconceptions in optimisation arise when students fail to accommodate new mathematical principles appropriately. Some learners remain at an assimilation stage, where they attempt to apply previously learned mathematical rules in inappropriate contexts, leading to conceptual errors. For instance, the data from student worksheets in this study revealed cases where students incorrectly used algebraic techniques rather than differentiation to find critical points in optimisation problems. Furthermore, the Action-Process-Object-Schema (APOS) theory, which extends Piaget's constructivist framework, provides additional insight into how students conceptualise mathematical ideas. APOS theory posits that students' progress through different levels of cognitive construction (Bintoro & Sukestiyarno, 2021; Waite-Stupiansky, 2022):

Action: Performing operations in a mechanical way without deeper understanding (e.g., simply applying derivative rules without comprehension).

Process: Recognising patterns and relationships (e.g., understanding that a derivative represents the rate of change).

Object: Seeing mathematical concepts as cohesive entities (e.g., recognising that differentiation and integration are interconnected processes in calculus).

Schema: Fully integrating concepts into a broader mathematical framework (e.g., applying optimisation techniques flexibly across different problem contexts).

Many of the misconceptions identified in this study indicate that students remain in the action stage, where they rely on rote memorisation rather than conceptual understanding. For example, some students correctly used differentiation to find maximum and minimum points but failed to interpret their significance within real-world contexts, suggesting incomplete cognitive development.

Results

The result of the study on Grade 12 students' misconceptions about the application of calculus in optimisation in a real-life problem was reported using a sample question gathered from the Grade 12 examination in South Africa. This sample question adopted for the process of data analysis is reported in Figure 1 and analysed in line with the suggested themes by the researchers.

Figure 1

Sample Question on Optimization

Sample question 1: *The volume of water in a tank, in litres, t weeks after a farmer starts measuring is given by* $V = -100t^2 + 200t + 2400$

1.1 After how many weeks was the volume at a maximum?

Explain how this maximum volume is found.

1.2 After how many weeks was the tank empty?

Give reasons for your working.

1.3 Determine the rate at which the volume is changing at $t = 5$*. Explain the importance of the + or - sign in your answer.*

Note: Sample question 1.1, 1.2, 1.3 is equal to 6.1, 6.2, 6.3 in the worksheet.

For data analysis, categorising and connecting strategies were employed. According to Maxwell and Miller (2008), data categories can be based on similarities and differences, which can be coded to identify the views of the participants without any confusion In addition to this, they argue that "coding categories are a means of sorting descriptive data so that the material bearing on a given topic can be physically separated from other data" (Maxwell & Miller, 2008, p. 465). For that reason, Creswell and Path (2018), and Watkins and Gioia (2015) support the use of categorising analysis strategies, viewing the class group as a case study and coding the data in a particular context, so that the contextual relationships of the coded data will not be lost (Maxwell & Miller, 2008). For clarity's sake, and to report the main aim and goal of the study, misconception is explained as a mathematics view that could be incorrect due to the faulty thinking or understanding suggested by the students when learning a mathematics-related topic. However, mathematical error is an act or product of wrong steps taken when reporting the view as conceived by a student and this may lead to a condition of ignorant or imprudent deviation from a mathematical behaviour (Neidorf., 2020).

Furthermore, we further analysed the result of the sample question which led to the selection of five Grade 12 students for learning optimisation. Table 1 shows that out of the 35 students that participated in the study, the result indicates that in Question 1.3, which is a 91% failure rate was recorded, while in Question 1.2, the failure rate was 89% and, lastly, Question 1.1 shows a 46% failure rate. This implies that there is a high percentage of misconceptions, which leads to a high rate of error among Grade 12 students when learning calculus. The results of the survey conducted using the worksheet are shown in Table 1 with their percentage.

Table 1

Analysis of the survey results per test item in percentages

Test item	No. of answers correct	% of answers correct	No. of answers partially correct	% of answers partially correct	No. of answers incorrect	% of answers incorrect
1.1	18	51	1	3	16	46
1.2	4	11	0	0	31	89
1.3	3	9	0	0	32	91

From Table 1, a high number of students are struggling with the modelling of some mathematics concept when learning of optimization in calculus. To probe into the causes of misconception and errors committed when learning optimization, we further selected the five Grade 12 students having an adequate knowledge of optimization in calculus. The study was guided by the research question**: What are the misconceptions generated by Grade 12 students when learning optimization in calculus related topics?** In answering this research question, some categories were stated and adopted in line with themes, as reported below:

Students' Misconceptions when Modelling Calculus into their Learning of Optimisation

Among Grade 12 students, optimisation is a course that helps these students in applying their mathematics ideas getting the maximum and minimum aspects of the calculated values in a real-life situation. This aspect of mathematics could be used in addressing some real-life problems in engineering, accounting, and medicine. Question 1 (Figure 1) focuses on a water tank, which is the area of specialisation of civil engineers, and this assists them in measuring the volume of water contained in a storage container.

Question 1 was used to test the students' ability to correctly apply (model) their knowledge on differential calculus and mathematical modelling to solve problems in the real-world situation. Also, the question tried to check the students' ability to integrate their knowledge on measurement and calculus in addressing the question of volume of water in a tank. This volume, in litres (l), is given by V = –100t2 + 200t + 2400, where t is the number of weeks after a farmer starts measuring. The first challenge we observed that some 14 students in this group failed to explain why V /= 0 had to be used to find time at maximum volume, while others used t = 0 for time when volume was at maximum or t = 0 for time when the tank was empty instead of V = 0. When required to find the rate of change of volume at t =

5, 11 of these learners used V(5) instead of V / (5). These misconceptions all relate to calculus intrinsic concepts of the meaning of the derivative of a function and its relationship with the original function. The details of their misconceptions are discussed later in different categories.

The Views of Grade 12 Students on the Application of Calculus in Optimisation

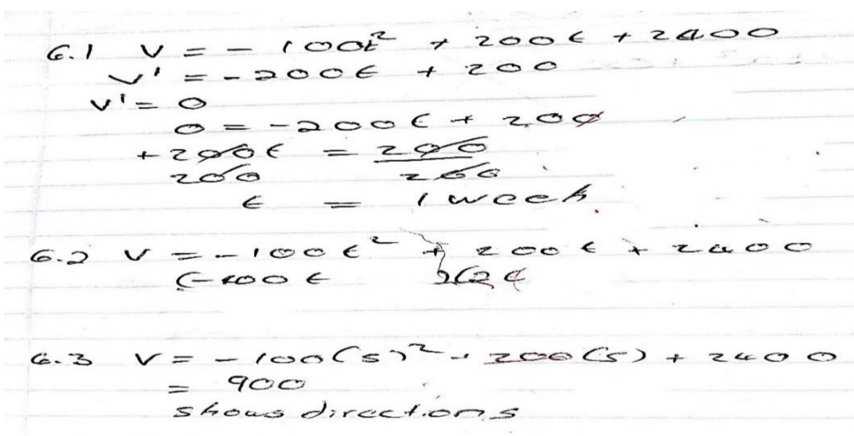
Among the 35 participants invited for the study, only the five participants with the best understanding on the learning of calculus and optimisation were selected for interviews. They were chosen after using the sample of their worksheet which was generated from the answers to the sample question which tested Grade 12 students' knowledge of calculus and optimisation. These five participants were given codes (L8, L9, L10, L11 and L12) to ensure their anonymity. The participants were requested to clarify their written solutions which are provided in the samples below.

Learner 8's (L8's) Misconceptions about the Application of Calculus in Optimization

Considering the sample question stated in Figure 1, we asked L8 to further explain his answer supplied in the worksheet, as indicated in figure 3.

Figure 2

Worksheet on Optimization for L8



6.1 $V = -100t^2 + 200t + 2400$

$V' = -200t + 200$

$V' = 0$

$0 = -200t + 200$

$\frac{+200t}{200} = \frac{200}{200}$

$t = 1$ week.

6.2 $V = -100t^2 + 200t + 2400$

6.3 $V = -100(5)^2 + 200(5) + 2400$

$= 900$

shows directions

In the sample of the question (Figure 2), one of the researchers that interviewed the participants questioned L8 to further understand the views on the modelling of optimization in a calculus related topic hereby continues with the following questions:

Researcher	Why did you use V =0 to answer a question on maximum volume?
L8:	The maximum is a turning point, so we equate derivative to zero.
Researcher:	In 1.2 you appear to have tried to factorise the expression for volume, why?
L8:	I wanted to solve for v=0 using factors for empty tank.
Researcher:	In 1.3, you used t=5 in the expression for volume of water in the tank, why?

L8: We are given the time t=5, then the rate of change in volume we must use the given time to find new volume.

Researcher: Explain what you mean by "show directions" in your answer.

L8: The + sign shows you are going right and – sign shows you are going left.

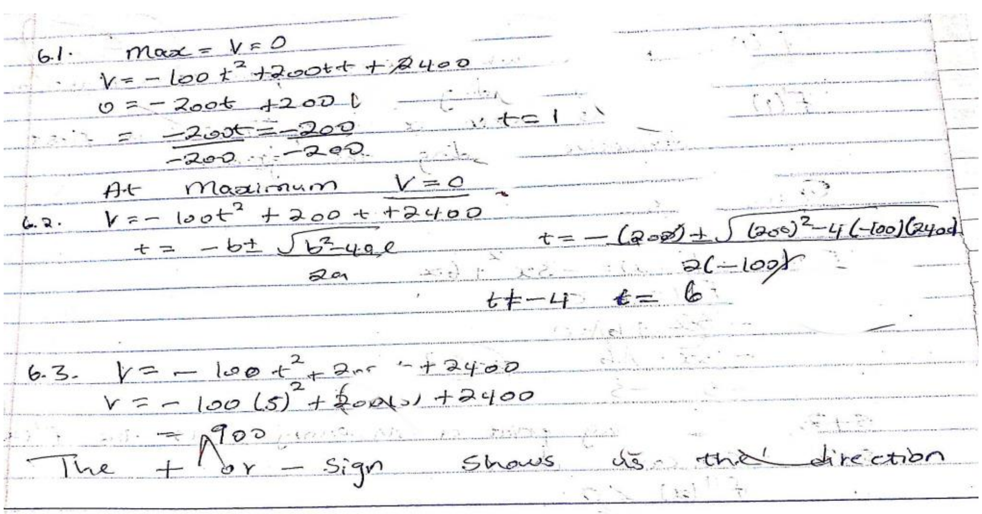
The view of participant L8 showed the correct understanding of maximum and minimum and how they related to the derivative of the function is that the derivative helps to find the time when volume is at its maximum. And the correct solution found in 1.1 seems to be a product of drill and practice rather than a conceptual understanding of the derivative as a rate of change. In Question 1.3, the learner failed to solve the same question on the rate of change of volume of water at t = 5, where the use of the derivative is required. The only logical explanation for L8's failure in question 1.3 is that his understanding of when to apply knowledge of derivatives is limited to maximum and minimum turning points. Secondly, L8 substituted t = 5 in the expression for volume, which is not a rate of change, showing that there is a misconception on what constitutes a rate of change as well as the units used to denote a rate of change of volume. These errors could reflect a shallow understanding of the derivative, where the knowledge is limited to textbook problems, but the learner is unable to apply it outside the classroom. On this note, the calculation shows that the participant possesses a low understanding of some concepts in optimisation, thereby resulting in the committing a conceptual error and some hidden errors. This corroborates the view of Herheim (2023), who argues that the misconceptions of the students when learning optimisation in calculus can be traced to poor recognition of the mathematical problem, understanding errors, connection errors, and mathematical rule errors, among many others. But the fact remains that all these forms of error should be overcome for proper modelling to take place.

The Views of Learner L9 on the Misconceptions exhibit in Calculus

The participants coded L9 was asked to explain his answer to Question 1 (Figure 1) as indicated in Figure 3 to help us further understand his views on the modelling of optimization in a calculus related topic.

Figure 3

Worksheet on Optimization for L9



6.1. max = v = 0

$v = -100t^2 + 200t + 2400$

$0 = -200t + 200$

$= \frac{-200t}{-200} = \frac{-200}{-200}$ $\therefore t = 1$

At Maximum $V = 0$

6.2. $v = -100t^2 + 200t + 2400$

$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $t = \frac{-(200) \pm \sqrt{(200)^2 - 4(-100)(2400)}}{2(-100)}$

$t = -4$ $t = 6$

6.3. $v = -100t^2 + 200t + 2400$

$v = -100(5)^2 + 200(5) + 2400$

$= 900$

The + or − sign shows us the direction

This is a section of the interview as reported below.

Researcher: Explain this statement which you wrote in 1.1 "max=v=0".

L9: When the volume is maximum, derivative v=0.

Student L9 was asked to explain his answer to 1.2, and L9 explained:

L9: If the tank is empty, it is equal to zero, so we solve equation for volume equal to zero using quadratic formula. Time cannot be negative, so the answer is t=6.

Researcher: In 1.3, you used t=5 in the expression for the volume of water in the tank, why?

L9: To get volume you substitute the time in expression of volume.

L9's first misconception is reflected in his failure to distinguish the function of volume from its derivative. The letter v cannot be used to denote both the volume and the derivative, as implied in this statement: "At maximum volume," it is the derivative when volume is maximum. The student paid little attention to the use of correct notation and the writing of mathematically correct statements, which could result from teaching and learning approaches where the focus was on the execution of procedures and arriving at answers with little conceptual understanding.

Secondly, the concept of a derivative as a measure of the rate of change of quantities has not been grasped by L9. The learner failed to differentiate the function from the derivative, which is failing to appreciate how the volume of water in the tank at a given time is different from the rate at which the volume is changing at that given time. This is evidence of a shallow understanding of the derivative concept and the broader contexts in which it is applicable. On this note, the participant had committed a generalisation error due to misapplication of concepts, and this eventually affected the result of the problem. This is in line with Cline and others who report that having a poor understanding of some mathematical rule can result in committing an error on a mathematical rule, which may lead to a general misconception (Cline et al., 2020).

Learner L10's Misconceptions on the Application of Calculus in Optimization

Using the answer to Question 1, L10 was asked to further explain the answer supplied in the worksheet, as indicated in Figure 4.

To further understand the views on the modelling of optimization in a calculus related topic, L10 was questioned about his answer which is displayed in Figure 4.

Researcher: Why did you use V /=0 to answer a question on maximum volume?

L10: When the word maximum or minimum is used, we equate derivative to zero.

Researcher: You used t=0 to find after how many weeks the tank was empty. Can you please explain?

L10 : At the start t=0 and the tank is empty, so I substitute t=0 in formula for volume.

Researcher: Your answer is V=2400, for an empty tank , how is this possible?

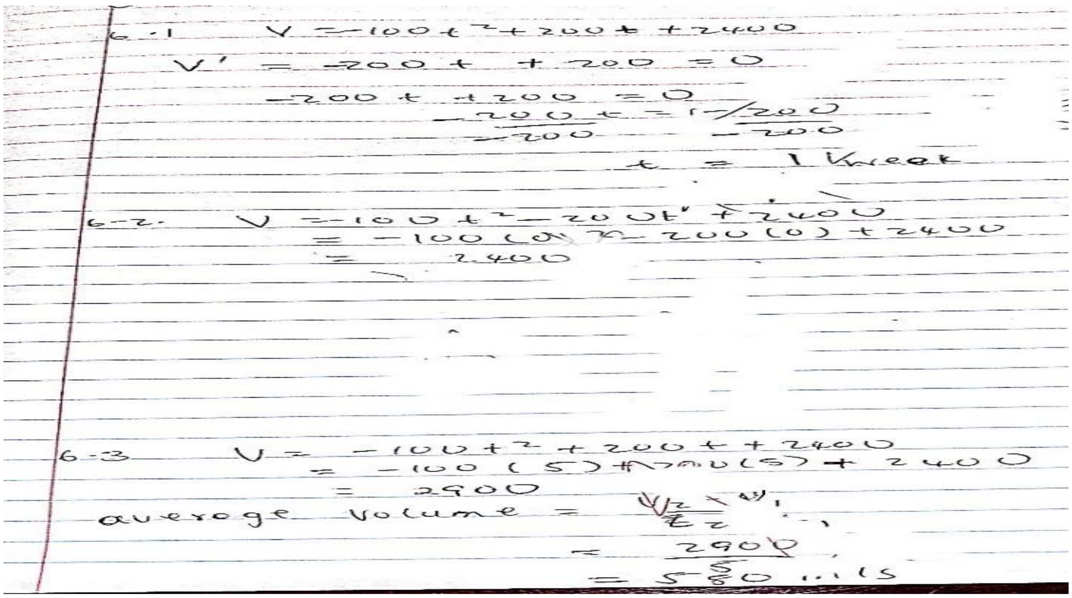
L10: The expression for volume is not correct, we must start with an empty tank when t=0.

Researcher: Can you explain your answer for the rate at which volume is changing in 1.3?

L10: The rate is given by the average change in volume from empty tank to the volume when t=5. I used (v_2-v_1)/(t_2-t_1) = (2900-0)/(5-0) because the starting volume and time are zero.

Figure 4

Worksheet on Optimization for L10



The fact that L10 emphasised the words maximum or minimum in deciding when to use derivatives could be an indication of memorisation of procedures with a shallow understanding of the derivative concept. However, the researcher had follow-up questions seeking clarification or elaboration on L10's answer when hesaid, "When the word maximum or minimum is used, we equate derivative to zero." The misconception on the derivative was confirmed in the learner's response in 1.3 when he was required to find the rate of change of volume at a given time. Instead of using derivatives to find the rate of change in volume, the learner found what he called "average volume," an indication that the learner does not fully comprehend the derivative concept in its different manifestations and when and how it should be used. When L10 uses V(5) to find the rate of change of volume when t = 5, instead of V / (5), this is a conceptual error on the rate of change where the learner failed to realise that rate of change of volume should be obtained from the derivative of the function, and rate of change is not just a measure of volume at a particular point in time but is an instantaneous measure of how the volume is changing with respect to time.

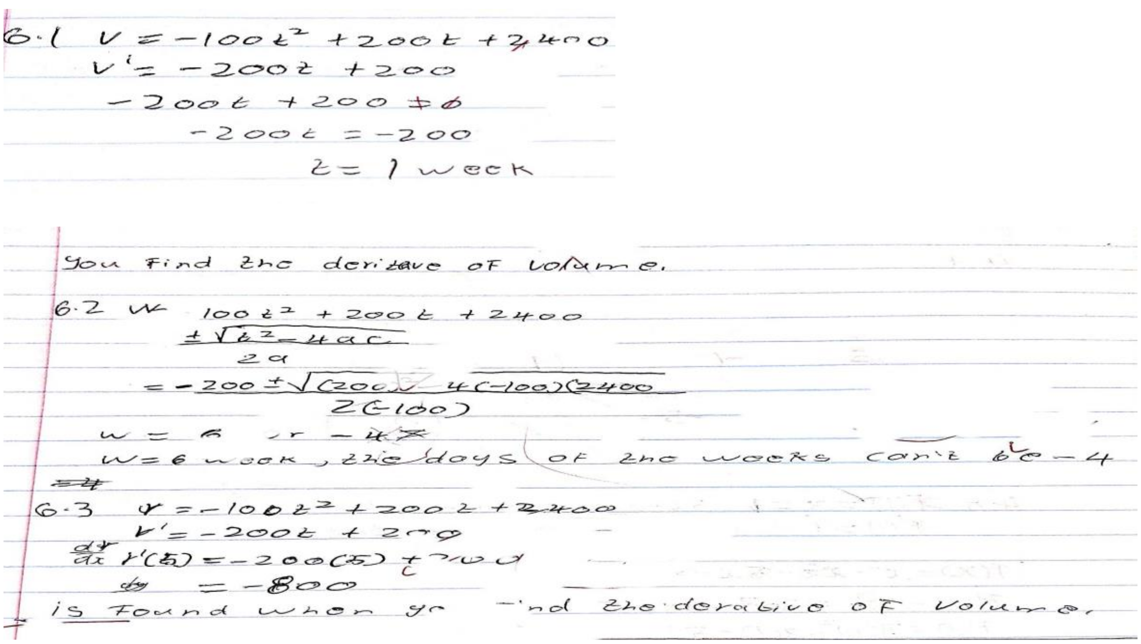
In answering the question on the time when the tank is empty, L10 used t = 0 instead of V = 0 and justified this by saying at the start t=0 and the tank is empty. The derivative of the given equation is (V'(t) = -200t + 200), the participant need to plug in (t = 5), which will give (V'(5) = -200(5) + 200 = -800) litres per week. The negative sign indicates that the volume is decreasing at the rate of 800 litres per week at (t = 5). The sign is important as it tells us whether the volume is increasing (+) or decreasing (-) over time. But this learner was so obsessed with executing procedures that he failed to see the contradiction in using time (t=0) to find time (no. of weeks) when the tank is empty and giving V = 2400 for an empty tank. This is evidence of a shallow understanding of what the learner was writing, as there is no room for reflection on what one is writing. L10 appears to have a limited understanding of the intrinsic precalculus concept of a derivative and hence struggles when presented with real-life contexts requiring the use of a derivative to resolve a given problem. This implies that the participant was ignorant of how to recognise and apply some aspects of calculus, thereby committing a conceptual error that eventually affected the process of modelling the mathematical ideas. The result of his views corroborates Cline and others who report that having a poor understanding of the mathematical problems in calculus could be caused by conceptual errors and misconceptions, which may lead to a visible misconception (Cline et al., 2020).

Learner 11's Misconceptions on the Application of Calculus in Optimization

Using the answer to Question 1, L11 was asked to further explain the answer supplied in the worksheet, as indicated in Figure 5

Figure 5

Worksheet on Optimisation for L11


6.1 $V = -100t^2 + 200t + 2400$
$V' = -200t + 200$
$-200t + 200 = 0$
$-200t = -200$
$t = 1$ week

You find the derivative of volume.
6.2 W $100t^2 + 200t + 2400$
$\frac{\pm\sqrt{b^2-4ac}}{2a}$
$= \frac{-200 \pm \sqrt{(200)^2 - 4(-100)(2400)}}{2(-100)}$
$w = 6$ or -4
$W = 6$ week, the days of the weeks can't be -4
6.3 $V = -100t^2 + 200t + 2400$
$V' = -200t + 200$
$\frac{dV}{dt}$ $V'(5) = -200(5) + 200$
$\frac{dV}{dt} = -800$
is found when you find the derivative of volume.

Figure 5 shows part of L11's answer to Question 1, to further understand the views on the modelling of optimization in a calculus related topic, L11 was questioned about his answer:

Researcher: Why did you use V /=0 to answer the question on maximum volume?

L11: To find time when volume is a maximum, we find derivative of volume and equate to zero to find time.

Researcher: Can you explain your working for finding time when the tank is empty? How is it linked to first statement w=-100t^2+200t+2400?

L11: The first statement shows water in the tank in each number of weeks, w. When the tank is empty, we get zero, so we use quadratic formula to solve equation. We reject negative answer for time as days cannot be – 4.

Researcher: Why did you find the derivative in 1.3?

L11: The derivative shows how volume is changing so we use it for rate of change of volume.

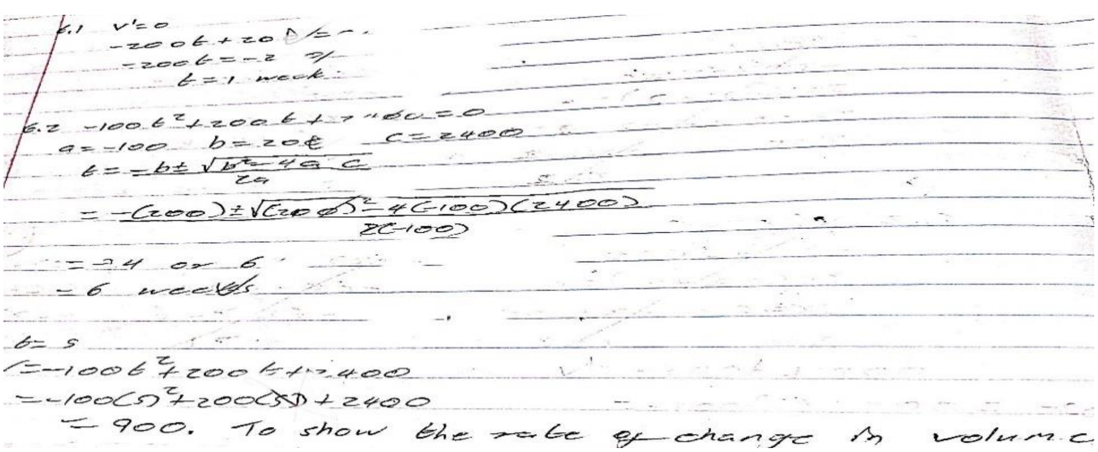
The findings and views of L11 demonstrate that he has the correct conceptual understanding of the derivative and can use the concept to resolve some real-life problems. However, in 1.2 there appears to be a misconception on how to present the working for finding time which is represented by t in v=-100t^2+200t+2400. The learner introduced w in place of V and the solution showed w=6 or-4. While the learner knew what he was trying to find, which is number of weeks, the statement w=-100t^2+200t+2400 was incorrect as it amounts to stating time (w) is equal to volume of water. The correct statement using w should be v=-100w^2+200w+2400 and the equation to be solved becomes 0=-100w^2+200w+2400. While the answer was correct, it was a result of a wrong hypothesis that time is equal to volume as implied by this statement w=-100t^2+200t+2400. The introduction of wrong mathematical concepts when learning calculus may affect the result of the problem, thereby causing a conceptual error (Thompson & Harel, 2021).

Learner 12's Misconceptions about the Application of Calculus in Optimisation

Using the answer to Question 1, L12 was asked to further explain the answer supplied in the worksheet, as indicated in Figure 6.

Figure 6

Worksheet on Optimisation for L12



1.1 $v' = 0$
$-200t + 200 = 0$
$-200t = -200$
$t = 1$ week

1.2 $-100t^2 + 200t + 2400 = 0$
$a = -100$ $b = 200$ $c = 2400$
$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
$= \frac{-(200) \pm \sqrt{(200)^2 - 4(-100)(2400)}}{2(-100)}$
$= -4$ or 6
$= 6$ weeks

$t = 5$
$V = -100t^2 + 200t + 2400$
$= -100(5)^2 + 200(5) + 2400$
$= 900$. To show the rate of change in volume

L12 was questioned about his answer to further understand his views on the modelling of optimization in a calculus related topic.

Researcher Why did you use V /=0 to answer the question on maximum volume?

L12: A maximum is a turning point and derivative is zero.

Researcher: Give reasons for your working to find time when the tank is empty in 1.2.

L12: When the tank is empty, volume is zero, you solve the quadratic equation for t. One of the answers is -4. It is not possible for number of weeks, so t=6 weeks.

Researcher: You used t=5 in formula for volume to find rate of change of volume. Why?

L12: When the time is changing, the volume is changing, so when t=5 the new volume is what I found using v=-100〖(5)〗^2+200(5)+2400=900 litres which shows the rate of change of volume.

The views of L12 show that he has a limited understanding of the derivative concept, which appears to be based on the concepts of maximum and minimum turning points. When the word maximum was used in 1.1, the learner had no problem identifying the derivative as the mathematical tool for addressing the given problem. However, in 1.3, when the question referred to “rate of change of volume of water," L12 failed to recognise that the question demanded the use of the derivative, as it is a measure of the rate of change of one quantity in relation to another. Hence, the learner had a misconception about when and how the derivative should be used in different contexts. In L12’s explanation on why he uses the rate of change when it is evident that he was failing to distinguish the measure of volume at a given instant from the rate at which the volume is changing, which is measured by the derivative. The learner’s knowledge could be classified as instrumental, as he was only able to use it in limited contexts. This could be a result of teaching and learning approaches that do not prioritise the development of conceptual knowledge in learners. This corroborates the views of Thompson and Harel, who argue that students learning calculus find it difficult to model their calculus ideas into their learning of optimisation because they do not understand how to model this concept in a realistic way, thereby causing a misconception and error (Thompson & Harel, 2021).

Discussion

This study was intended to test the students’ ability to correctly apply their knowledge of differential calculus and mathematical modelling to solve problems in the real world. It checked the students’ ability to integrate their knowledge of calculus into measurements and optimisation in line with the Grade 12 CAPS curriculum. The envisaged misconceptions were on the interpretation of ‘an empty tank’ where learners could assume that the tank was empty instead of being used. The other misconception could be on rate of change, where students might fail to interpret the derivative as measuring the rate of change of the volume of water in the tank. These misconceptions are discussed in line with the views of the Grade 12 students learning calculus and optimisation:

- The conceptual errors and misconceptions about optimisation in calculus
- Generalisation errors and misconceptions about optimisation in calculus
- Language errors and misconception about optimisation in calculus
- Other hidden errors and misconceptions about optimisation in calculus

Sample Question 1 (Figure 1) was intended to test the Grade 12 students' ability to correctly apply their knowledge of differential calculus and mathematical modelling to solve problems in the real world of optimisation. It also attempted to check the students' ability to integrate their knowledge of optimisation, measurement, and calculus as concepts. The views of the participants are reported and discussed in relation to common misconceptions among Grade 12 students in a high school in South Africa. All these were reported to address the problem of poor performance among Grade 12 students' when writing the National School Certificate (NSC) examination in South Africa.

Conceptual Errors and Misconceptions about Optimisation in Calculus

The results gathered from the data for this research highlight students' challenges in distinguishing volume as a function of time in examples like V = –100t2 + 200t + 2400 and the derivative V/= –200t + 200, which is a measure of the rate of change of volume with respect to time. It was discovered that some students in this group of fourteen were unable to explain why V/= 0 but had to find the time at maximum volume. Others used t = 0 for the time when the volume is at maximum or t = 0 for the time when the tank is empty instead of V = 0, as L10 did in Figure 4. When required to find the rate of change of volume at t = 5, L12 in Figure 6 used V(5) instead of t/(5). This shows that some Grade 12 students exhibit some conceptual errors, which result in some misconceptions, as reported by the result. Therefore, one could argue that the students have poor understanding of the concepts involved in optimisation, and thereby they could not get the right stages to follow when modelling their ideas into the learning of optimisation. This is in line with some researchers who argue that some issues like conceptual errors, poor interpretation, and application errors affect the poor modelling of mathematical ideas, which eventually turns to misconception if not properly handled (Cline et al., 2020; Herheim, 2023).

In a nutshell, conceptual errors occur when students misunderstand fundamental calculus concepts. The findings revealed that many students struggled with distinguishing between a function and its derivative, leading to errors in problem-solving. For instance, some students equated the function with its derivative, assuming both represented the same mathematical concept. This resulted in misinterpretations when identifying maximum and minimum values. One common misconception observed was the belief that setting would yield the maximum volume, rather than correctly equating to find critical points.

To address conceptual errors, targeted interventions should emphasize:

- Explicit teaching of function versus derivative distinctions.
- Visual representations and graphical analysis of optimisation problems.
- Concept-based assessments to reinforce understanding beyond procedural calculations.

Generalisation Errors and Misconceptions about Optimisation in Calculus

Apart from the poor conceptual understanding that led to a conceptual error, as observed in each participant's worksheet, we further observed that there were generalisation or transfer errors. This form of error is committed by students when modelling some concepts on derivatives and how they relate to the rate of change of water volume.

This evidence, as found in the worksheet, shows that these students have not mastered what Herheim (2023) refers to as relational understanding and what Cline et al. (2020) refer to as conceptual knowledge of the derivative concept and how it is applied to successfully resolve real-life problems through mathematical modelling. When learners have only mastered an instrumental understanding of differential calculus concepts, they eventually struggle when required to apply these concepts in solving real-life problems, sometimes making what are termed conceptual and transfer errors, as is the case here, which could affect the result (Makonye & Luneta, 2014). Many researchers believe that the major source of conceptual, generalisation, or transfer errors resulting from students relying on instrumental knowledge of concepts to resolve problems in broader contexts is due to poor teaching strategies that emphasise drill and procedures as well as computational skills (Bezuidenhout, 2001; Toh, 2007). The generalisation error is applicable to four participants that participated in the interview section of the study—L8, L9, L10, and L11. Thus, the study corroborates the view of Makonye and Luneta (2014), who argue that during the learning of optimisation in calculus, the tendency to commit a general error is very high due to a low level of understanding of some conceptual aspects.

In a nutshell, generalisation errors arise when students incorrectly apply previously learned rules to new contexts without considering their limitations. The data analysis showed that some students misapplied algebraic techniques if a single approach would work for all types of optimisation problems. For example, a few students used the quadratic formula indiscriminately without verifying whether the given function was quadratic in form. Others assumed that differentiation always led directly to a solution without considering the need for interpretation of results within a real-world context.

To mitigate generalisation errors, instructional strategies should include:

- Encouraging flexible problem-solving approaches.
- Teaching students to assess the structure of mathematical problems before applying known methods.
- Providing varied problem sets that require adaptive reasoning.

Language Errors and Misconceptions about Optimisation in Calculus

Mathematics on its own is a subject that comes with its own language because it has its own symbols, alphabet, and numerical values with their meanings and signs, and it requires a better understanding of some skills. But the fact remains that some students learning mathematics, particularly in calculus, do not understand the language required or when to apply it appropriately (Alt et al., 2014). From the data supplied by L8, L10, and L11, it could be said that most students misapply some aspects of calculus into their learning because they find it difficult to interpret the given question. This corroborates Alt et al. (2014) who argue that the mathematical difficulties of children who are taking English as a non-mother tongue language may stem from a difficulty of interpretation. Therefore, students with the opportunity of having English as their mother tongue may have a better understanding of calculus and optimisation compared to a non-English speaker (Alt et al., 2014). This is because the language errors that may be committed may be reduced, thereby resulting in a low misconception. But students with a low level of understanding of English may have a high level of misconception. This implies that participants L8, L9, L10, and L12 had a low level of understanding of English as a means of communication in mathematics.

In a nutshell, Language-related errors were observed when students misinterpreted mathematical terminology, symbols, and instructions. The study found that learners who were not proficient in English often struggled with understanding the wording of optimisation problems. For instance, terms such as "rate of change" were frequently misinterpreted, leading to confusion between the derivative and the function itself. Furthermore, notation errors occurred when students improperly used symbols such as "=" instead of "" in approximation scenarios, or misread question prompts due to language barriers.

To reduce language-related errors, the following measures should be considered:

- Incorporating bilingual instructional materials where possible.
- Reinforcing mathematical terminology through repeated exposure and contextual examples.
- Using visual aids and interactive teaching strategies to supplement textual explanations.

Other Hidden Errors and Misconceptions about Optimisation in Calculus

From the data collected from the participants, it has been clearly shown that some of the participants encountered challenges when learning optimisation, and this resulted in errors and misconceptions that, if not corrected, could be a source of mistake among them. For instance, some participants like L8, L9, and L11 were found to be committing an ignorance of rule restrictions error. This is an error made because of the overgeneralisation of some mathematics procedures among the students. Another error is the incomplete application of rules, which resulted in students' failure to apply the mathematical rules appropriate for the teaching of optimisation. Finally, there is the hypothesis error, which is an error generated by forming the wrong hypothesis about concepts that eventually affects the result (Jameson et al., 2023). In summary, it is important to note that the process of committing misconceptions cannot be possible without considering nature, nurture, and experience; therefore, the misapplication of APOS theory has resulted in unnecessary errors committed among Grade 12 students (Brijlall & Ndlovu, 2013).

In a nutshell, hidden errors refer to mistakes that are less obvious and often go unnoticed but significantly impact problem-solving accuracy. These errors include:

- Incorrect assumptions about real-world constraints in optimisation problems.
- Errors in interpreting graphical representations.
- Incomplete or skipped procedural steps, leading to incorrect solutions.

For example, some students failed to recognize that negative time values in their calculations were not meaningful in a real-world context. Others neglected the second derivative test when confirming whether a critical point was a maximum or minimum, leading to incorrect conclusions.

To address hidden errors, the following approaches should be implemented:

- Encouraging students to verbalise their reasoning and justify their solutions.
- Providing step-by-step feedback on problem-solving approaches.
- Conducting peer review exercises to help students.

Conclusion

The study investigated the Grade 12 students' misconceptions when modelling their calculus ideas into their learning of optimisation in a real-life situation. The results showed that the learners learning optimisation in their NSC examination agree on the misconceptions they have when learning and doing optimisation in calculus. This was reported through their worksheet and oral responses to interview questions. During the process of their learning, students made some errors and misconceptions, and these include systematic errors that occur when learning calculus-related topics such as optimisation limits and derivatives; when learning appropriate notation and formulas; algebraic manipulations; graphical application of calculus; and the use of calculus in optimisation. Evidence from samples and interviews of learner solutions indicated that learners had only mastered an instrumental understanding of the calculus concepts that were taught and had no problems with executing procedures of optimisation but lacked conceptual understanding of the calculus concepts with which they were dealing. This lack of conceptual knowledge or relational understanding of optimisation errors, from conceptual errors to general errors, which also include language errors, interpretation, and modelling errors, which are regarded as other hidden errors, had contributed to misconceptions among Grade 12 students when learning optimisation in a high school in Limpopo, South Africa. Therefore, students should have enough knowledge of the relational understanding of some mathematical aspects like optimisation.

Finally, this study suggests that the increase in student understanding when working on optimisation in calculus, could be achieved if the students have a better understanding of the major concept of optimisation by having in-depth knowledge of it. As practising teachers, Fumador and Agyei (2018) based their recommendations on how these misconceptions could be reduced among Grade 12 students when teaching optimisation, by suggesting that teachers should give regular warnings to the students to avoid misconceptions; the optimisation concept should be taught repeatedly; and recognising errors and misconceptions when diagnosing conflicts in their teaching and learning approach could also be of assistance.

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