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Fostering the Development of Young Students' Analytical Thinking by use of a Problem-solving Method

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Abstract: The analytical thinking level of secondary school students in the Netherlands has been on the decline for the past few years, with grave consequences for the mathematical level of these students, who often struggle later on in their academic careers. In particular, these students showcase underdeveloped basic skills such as critical reading and critical thinking, vital to many subjects in secondary school. Problem-solving methods have been utilised widely across the literature to foster both academic skills and the performance of students. In this research, a problem-solving method, inspired by Polya's four step method, is introduced and extended to include a reflection part (inside phase) to help students foster and develop their analytical thinking. Qualitative findings from a study conducted with K–8 students are reported and discussed to determine the degree to which the methodology helped these students develop their analytical thinking compared to a parallel class of K–8 students.

Keywords: Analytical thinking; mathematics teaching; Polya's method; problem solving; reflection as a learning method
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Introduction

In recent years it has become increasingly apparent that the level of so-called 'basic skills' (SLO, 2022) attained by secondary school students in the Netherlands is on the decline. Especially critical reading, critical thinking and arithmetic, skills which play a role in almost every subject in secondary school, and which are important for the fostering and the development of analytical thinking, seem to remain underdeveloped (Inspectie van het Onderwijs, 2022; Inspectie van het Onderwijs, 2023; Inspectie van het Onderwijs, 2024). As a consequence, students do not make progress with basic skills such as reading and counting. In 2023, a sample of 225 secondary schools revealed that the quality judgement of over 20% of these schools was deemed insufficient with regards to the students' ability to read and count (Inspectie van het Onderwijs, 2024). The low level of critical thinking directly impacts the understanding and complexity achievable for the subject of mathematics, with dire consequences for the mathematical prowess and future academic career of students: reports from 2022 mention that more and more Dutch secondary school students pass their final centralised exams and proceed to tertiary education in spite of a general decline in the level of analytical thinking and mathematical skills (Inspectie van het Onderwijs, 2022). The poor mathematical basis acquired during students' formative years is, therefore, carried throughout subsequent studies, leading students to often struggle with the proposed mathematics courses in their further education. This, in turn, has influence on the drop-out rates of these students as correlation has been found to exist between the students' results in their final national secondary school exam, their GPA during their first year of university and their eventual graduation from a Bachelor study programme (Hernandez-Martinez, 2016; Pinto & Koichu, 2023; Rach & Heinze, 2017; de Winter & Dodou, 2011). More generally, the PISA 2022 study revealed a trend across the EU by which 30% of secondary school students do not reach a minimum proficiency level in mathematics (European Commission, Directorate-General for Education, Youth, Sport and Culture, 2024). However, this trend is particularly felt in the Netherlands, where students show less interest in mathematics. Compared to their peers in other countries, Dutch students find mathematics less important and put less effort into its study while concurrently displaying more confidence in their knowledge (SLO, 2024).

In this paper we describe a problem-solving method, which finds its roots within Polya's four step method, aimed specifically at the fostering of analytical thinking. In particular, this research set out to determine the extent to which the developed and introduced problem-solving method improved the analytical thinking of young students. This was done by qualitatively analysing the work of students as pertaining to three different core objectives of the Dutch curriculum (introduced as part of the theoretical framework). These three goals define themselves research questions aimed at understanding to what extend each of these goals is fostered and developed through the use of the introduced method. The problem-solving method was introduced in 2023 in a K–8 class with the aim of determining its effect on the development of students' analytical thinking. We report our findings as qualitatively measured and observed and compare these to observations relative to a parallel K–8 class not subject to the method.

Theoretical Framework

The low level of analytical thinking hinders students' understanding, who then often learn mathematics in an instrumental way rather than a relational one (Skemp, 2006). The absence of relational understanding, in turn, also lowers the level of analytical thinking as both skills pertain to the interest in the workings behind processes and in the connections between subjects. In fact, we understand analytical thinking as the ability to be able to carry out the following thinking tasks upon a given problem (Amer, 2005):

- divide the given problem into main and sub-questions in order to be able to tackle the problem in a structured fashion;
- collect, organise and (possibly) visualise the data given in the problem;
- make connections between the given data of the problem;
- if possible, make connections between parts of the problem and own existing knowledge;
- make estimates for the answer required by the problem based on the given data;
- critically examine and analyse one's own process and the answer found so as to be able to indicate strengths, weaknesses and any present illogic;
- critically examine the process and the answer found by others so as to be able to indicate strengths, weaknesses and any present illogicality.

As the definition of analytical thinking is a broad one that includes a variety of skills, the development of which may be difficult to test or account for, this research focuses on the aspects of analytical thinking that are also specifically mentioned as part of the core objectives for 'basic skills' the Dutch secondary school curriculum aims at developing. In particular, achieving the development of analytical thinking is further approached in this research as the attainment by the students of the following three core objectives (SLO, 2022):

- core objective 19 (*The student learns to use appropriate mathematical language for the organization of one's own thinking and for explanation to others and learns to understand the mathematical language of others*);

- core objective 23 (*The student learns to make exact and approximate calculations and learns to reason based on insight into accuracy, order of magnitude, and margins that are appropriate for a given situation*);
- core objective 27 (*The student learns to systematically describe, organize and visualize data and learns to critically assess data, representations and conclusions*).

Analytical thinking as understood through core objectives 19, 23 and 27 is fostered and developed in this research by use of a problem-solving method, the steps and phases of which address various aspects of these core objectives.

Analytical thinking and problem solving

The idea of teaching analytical thinking skills to young students is one that is very reoccurring both in the literature and in practice. Various schools, coaches and teachers (and even parents with a background within the educational sector) work under the assumption that intelligence need not be static and that young children can learn skills such as analytical thinking and problem solving with the appropriate guidance. Indeed, analytical thinking is best developed through discussion and reasoning about the material, tools which students hardly make use of in the absence of the teacher (Murphy et al., 2014). This means that thinking and sharing discoveries should be promoted within the classroom and that the teacher should take an active role in fostering this process. An example of this is Morningside Academy, which has been implementing effective guidance in developing analytical thinking skills in the form of a variation on TAPS ('Think Aloud Paired Problem Solving') (Robbins, 2011). Here students are taught to make use of 'private speech' (Berk, 1994). Furthermore, the use of modules regarding problem solving that includes the steps 'identify, define, explore, act, and look back' promote the development of analytical thinking (Karenina et al., 2020). These steps are, in part, recognisable in Polya's four step method, which includes the steps 'understanding the problem, devising a plan, carrying out the plan, and looking back' (Polya, 2014). What makes Polya's method unique among the other problem-solving methods is the fact that not only has previous research shown Polya's method positively impacts students' attitude towards mathematics (Yapatang & Polyiem, 2022), but that it also performs better than other educational methods (George & Enefu, 2019) and promotes relational understanding and understanding of conceptual and algorithmic problems (Bilgin, 2006; Heutinck, 2022). Polya's method is, additionally, often recommended as a classroom tool (Lee, 2017; Obiano & Parangat, 2023) that is also utilised as measure of mathematical thinking ability (Nurtamam & Jannah, 2024) and has been found to be applicable to students in tertiary education as well (Litvak & Weedage, 2023). However, this method has drawbacks as well. It is the experience of the author and of colleagues that the second step of Polya's method, 'devising a plan', very often poses a challenge to students rather than helping them find their way through a posed problem. Students experience this step as awkward, often get stuck and may subsequently decide to either skip the step or abandon the assignment entirely (Muniri & Choirudin, 2022). Furthermore, if step 2 is carried out incorrectly, this may further impact the ability of students to correctly solve the problem, yielding incorrect answers that the students are then not able to properly evaluate (Yayuk & Husamah, 2020). This is partly also due to the fact that devising a plan is made feasible by a wide variety of learning experiences (Hadi & Radiyatul, 2014), which young students do not yet possess. Creativity to this avail is often not fostered in schools. Polya's method also does not provide enough in class guidance for students to be able to learn how to properly set up a plan and to reflect on one's own work and, more in general, does not provide students with the possibility to reflect in depth on both the positive and negative aspects of their work (Mazur, 2016; Yayuk & Husamah, 2020). Reflection is a key component of critical thinking and has been found to influence students' mathematical self-image, ultimately benefitting their success (Colley et al., 2012). Spending time within the classroom is a must to allow students to really develop reflection skills (Marzano & Miedema, 2018), which is why a different problem-solving method is proposed and utilised here.

Methods

The study portrayed in this paper was conducted in 2023 in a secondary school in the Netherlands. With the consent of school coordinators, two K–8 classes with identical curriculum (hence parallel classes) as well as their respective mathematics teachers were asked to participate in the study: one class as research group and the other as control group. The research group consisted of 18 students, while the control group consisted of 20 students. Both classes were so-called 'sport classes', meaning that, even more so than any regular parallel classes, these two K–8 consisted of children with similar interests, allowing for a more direct comparison between the two groups during and after the study detailed in this paper. No further biodata was utilised so as to minimise bias and impact upon the students.

The choice for K–8 was determined by two main factors: curriculum and positioning within the Dutch school system. In fact, within the Dutch school system, K–8 corresponds to the second year for students in secondary education, meaning that students at this stage would gain great benefit from fostering the development of their analytical thinking throughout the many formative years ahead of them (McClelland et al., 2006; Myers & Conner, 1992). These students also do not have to be held to the strict exam regulations required in later years, making it possible to adjust their planning to allow their participation in the study. Additionally, they are not as experienced within the school system and have, therefore, not yet grown the belief that the subject of mathematics requires little to no written text. Concurrently, K–8 students work through mathematical topics that allow for problem formulations requiring multiple steps before reaching the answer and that are, therefore, better suited to the introduced problem-solving method as well as better suited to let the requirements of the considered core objectives shine through in the work of the students. The introduced assignments, in fact, were designed to face students with a productive struggle, a significant activity for students to be able to learn mathematics in a relational fashion (Granberg, 2016; Hiebert & Grouws, 2007).

Problem-solving method

The problem-solving method utilised within this study requires students to solve problems according to the following three steps:

1. **Preparation**: students are asked to read through the given problem and to describe in own words what the goal of the task is. Students must also give a summary of the data available to them. Utilising this summary, if necessary, students must compile the data into a sketch to better describe the situation. In addition, students are here asked to indicate which pieces of data or information are necessary for the final

answer to be achieved.

Guiding questions are provided to help the students along the way, such as:

- What do you have to calculate/prove?
- What pieces of information do you need to find the answer?

2. **Calculations**: throughout step 1, students have indicated their starting point in the problem and have outlined the finish line. In this step, students are asked to follow up on step 1 by performing the calculations or steps of a proof in an orderly manner. Students must write out each calculation clearly by placing each new step on a new line and by numbering the lines containing calculations. Students must further indicate in a few words what they are doing and why they are doing it as they will need this information for step 3. It may happen that students get stuck and do not find a solution to the given problem. In that case, students are invited to take a break and attempt the problem later, before reaching the definite conclusion that they cannot achieve a solution. If the students still cannot find a solution, they are asked to clearly indicate this and to continue to step 3.
3. **Explanation and Reflection**: if students have managed to find a solution to the problem, they are asked to explain each step of their calculation in full sentences using the same numbering as in step 2. Guiding questions are again provided to help students along the way such as:
   - What did you do in this line?
   - Why are you taking this step and what does it achieve?
   - Why may you make this calculation?

   If students are unable to reach a conclusion during this step, they are instead asked to first explain the steps they did manage to work out (line by line) and are then invited to answer the following questions in full sentences:
   - What have you achieved so far?
   - Where did you get stuck?
   - Which pieces of information were you missing in order to be able to continue?
   - How could you still find or calculate these missing pieces of information?
   - What answer would you have expected? Give an indication of the answer you would have expected to find and explain why you think this might be the solution.

Students are not required to make a plan, ensuring they are not hindered in their process, and are instead asked to think about the information necessary to the problem. Through step 1, the focus is then placed upon what can and should be done rather than what could be done to find the solution, addressing the issues brought about by the second step of Polya's method. Step 1 addresses core objective 27 by having students explicitly describe, organise and visualise the data at their disposal. Step 2 addresses both core objective 19 and 23 as it requires students to write down both their calculations and thought process in detail, promoting the development of their use of appropriate mathematical language. Step 3 particularly addresses core objective 23 as students must reason and evaluate the correctness of their answer based on insight and most otherwise reason what an appropriate solution might be following the problem constraints.

The problem-solving method includes a set of instructions for the students detailing the three steps explained above as well as a fully worked-out example for the students to read through and see what exactly is expected of them in each of the steps.

Implementation

This study was conducted while both K–8 classes were working through chapter 8 ('Volume and Enlargement') of their book. Across the four weeks allocated to work through this chapter before the test, the students of the research group were given three different problem sets, each containing one assignment, to be solved according to the problem-solving method as reported above. Additionally for this group, of the four weekly hours of mathematics, one was devoted entirely, for the purpose of this study, to the discussion and reflection of the problem set of the week (the very first of these hours was dedicated to the introduction of the study and the problem-solving method). Effectively, the research group took part in the study through two separate phases: one outside of the classroom (*outside phase*) and one inside of it (*inside phase*). The control group too received the problem sets during this time, but received no further instruction on how to solve them, so as to be able to compare the difference between the two groups both across the problem sets and the final test.

Aside from implementing the problem-solving method within the control group, at the end of this study semi-structured interviews were held with students from both K–8 classes. The test results of both classes were further also utilised to compare the research group to the control group.

Outside phase

During the outside phase, students were handed take-home problem sheets. These had to be solved withing one week, after which the inside phase would take place. The research group was specifically instructed to solve each exercise contained in the problem sheet according to the problem-solving method reported in this paper. Students were allowed to work together as long as each student still produced their own individual solution just as they were allowed to use the internet and their book to search for formulas or inspiration. Furthermore, the research group was instructed to write their solutions on sheets of paper separate from the students' usual homework notebook. This was to ensure that the written up solutions could be handed in and collected to form a student's portfolio through which their progress could be tracked. The control group also took part in this phase, but was given the problem sheets as simple additional homework which they had to hand in to their teacher after a week. These students too could work together, but had to each produce an individual solution. Their teacher would then discuss the solution with them classically at the whiteboard. For both groups the problem sheets were compulsory and the handed in solutions formed a portfolio of the students' progress. The teachers of both groups were instructed not to answer questions about the take-home problem sheets so as to minimise the impact of different teaching styles. Collaboration among students was allowed as policing this would not have been possible outside of the classroom. Requiring individual solutions to be handed in, however, still allowed for observation of the students' individual thought process.

The outside phase took place three separate times throughout the study, for a total of three separate problem sheets. These assignments may be found (translated from the original Dutch) in the appendix of this paper.

Inside phase

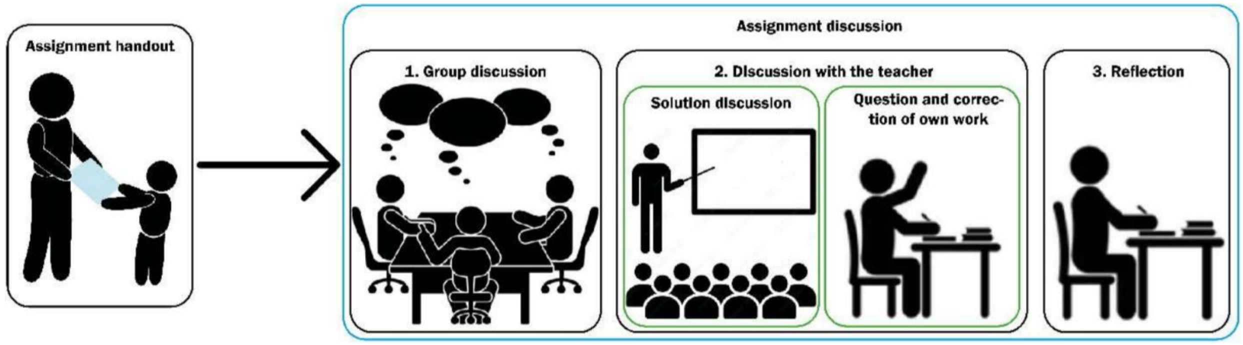
The inside phase was unique to students from the research group and required them to bring their (partial) solution to the problem set of the week to class. During this phase, students were first divided into small groups consisting of three to four students and given time to discuss and compare their solutions. The groups were different every time and decided by their teacher. This gave every student the opportunity to actively talk about mathematics, their thought process and to discover that of a variety of their peers. For students who did not find a solution to the problem set, this was an opportunity to ask fellow students how a solution could have been found, providing them insight into the steps they might have overlooked. These group discussions allow for the development of core objective 19 as students must use correct mathematical language to find a common ground upon which to communicate their different solutions. During the first instance of this phase, the teacher served here a supporting role, guiding discussion when necessary, as students had never had to work and discuss in this fashion before. By the end of the discussion, the teacher discussed the assignment centrally on the whiteboard utilising the input from the various groups of students, so that a solution was shown with the active involvement of all students, who had to use this time to also correct any mistakes in their approach and resolve any doubts by asking questions. Finally, students were handed out a reflection sheet (with contents unique to the session and problem set). On this sheet, students had to complete a series of tasks. They had to indicate which tools they had used to solve the exercise (such as their book, the internet or perhaps a fellow student), what had gone well (which steps had been carried out correctly or what were they able to notice about the problem that aided finding a solution), the type of mistakes they made (such as misunderstanding what they had to do or computational mistakes) and what they had learnt from the session. They were further asked to state their understanding of the mathematical topics involved in the assignment by indicating per listed topic on the sheet whether their understanding was 'good', 'required revision' or 'required to be studied again'. Finally, students were asked to formulate a few concrete steps they were going to take in order to improve their understanding of the topics for them still in progress. Here too, during the first instance of this phase, the teacher provided support when necessary, guiding students to see the good in their work and helping them formulate concrete and achievable plans. These filled out reflection sheets were also collected into the students' portfolios and addressed the lack of reflection present in Polya's method.

Just as the outside phase, the inside phase (including group discussion and reflection sheet) took place three separate times, one relative to each problem sheet. An example of one of the utilised reflection sheets is included (translated from the original Dutch) in the appendix of this paper.

The complete research methodology as introduced within the research group, including both the outside and inside phases and the different aspects of the inside phase, is schematically represented in Figure 1.

Figure 1

Schematic visualisation of the research method



Interviews

After the conclusion of both the chapter and the test, semi-structured interviews were held with three students per group. During these interviews, all six students were asked to solve a mathematical problem while thinking out loud about their steps. They could, at any point, ask the researcher questions and clarifications as well as help in case they got stuck. Once the problem had been discussed, students were asked about their thoughts on the quality of the assignments and the time investment necessary for them to complete each one. Research group students were additionally asked to give their thoughts on the problem-solving method, whether this had helped them develop a deeper understanding of the material and the effect they thought this had on them and their competency.

Data collection

This study collected data through the use of a variety of tools: problem sheets, reflection sheets, interviews and one exam at the end of the chapter.

Problem sheets

The utilised problem sheets were designed as a set that would encompass all learning goals pertaining to the material of chapter 8. These problems were designed to cater to all three core objectives as well as the learning goals, which were outlined together with the teachers of the two classes. The designed problems were reviewed in light of the core objectives and learning goals by both the teachers involved in this study as well as two researchers and educational experts from the University of Twente in the Netherlands. The problems were formulated to provide as little guidance as possible: no in between questions were asked that would point students in the correct direction of the final solution so as to highlight the impact of the method. To further ensure that the impact of the method could become visible, both teachers were specifically instructed not to answer questions regarding these problem sheets. The instruction sheet given to the research group was factually the only item creating a difference between the two groups of students. This sheet was extensively reviewed by both teachers and the two educational experts so that its formulation would be crystal clear for students: this was done so as to eliminate the confounding variable of students possibly not understanding what was being asked of them.

Reflection sheets

The utilised sheets were designed to have students from the research group reflect on their work and knowledge based on the group discussion taking place during the inside phase. The sheets were evaluated to refine wording and ensure clarity of requests, however, as the students had never before been asked to reflect on their own work in this fashion, support and clarifications were necessary from their teacher.

Interviews

Semi-structured interviews were held and recorded with a set of students from either class. The questions utilised in the interview were formulated and reviewed so as ensure clear and neutral formulation. The teachers from both classes chose the candidates for the interviews based on the requirements that per class at least one 'strong', one 'weak', one self-assured and one insecure student would participate. This was to ensure that each group of students within the class would be represented, but some bias may have been introduced.

Data analysis

The collected data was qualitatively analysed through use of coding. In particular, the solutions to the problem sheets handed in by students of both groups, the reflection sheets and the transcripts of the interviews were coded with the goal of individuating episodes showcasing elements from each of the three core objectives. Additionally, the solutions to the problem sheets were analysed as portfolios so as to be able to do a descriptive analysis of the differences (be it improvements or else) over time. The coding was carried out by the researcher and one educational expert. Due to this limitation, data was only considered as valid proof if both coders agreed on its interpretation.

Results

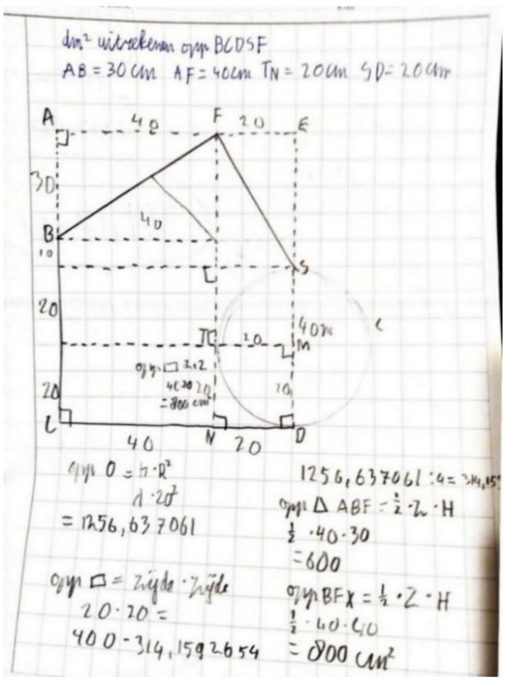
The results of this study comprise of multiple parts, as the findings relative to the problem sets, interviews and the test were taken into account and qualitatively assessed for both groups. For the research group, additionally, the findings and experiences relative to the inside phase of the research method are also reported.
Overall, the findings for every individual part rather promising regarding the positive impact of the introduced methodology for the fostering of the development of analytical thinking for students.

Problem sets

While some sort of precaution was taken by choosing a K–8 class as research group, students were, in line with expectations, at times still not enthusiastic about the amount of writing required for the problem-solving method. Some students indicated that this might have been because no grade was attached to these assignments. Especially students who were very confident in their mathematical abilities and skills did not feel inclined to work particularly hard on these problems as, in spite of the explained theoretical benefit the problem-solving method could have for them and their understanding of the material, they felt the difficulty of the assignments would be more impactful than the problem-solving method itself. Even these students, however, always provided a sketch of the situation so that it would at least be clear which steps were utilised to calculate what, indicating a clear development in core objective 27. Additionally, these students started applying more and more steps from the problem-solving method throughout the study and the various problem sets. They indicated, to this avail, that this allowed them to have more in depth conversations and discussions about their work and solution during the inside phase.

Figure 2

Page 1 of the solution to problem set 1 by a confident student (research group).

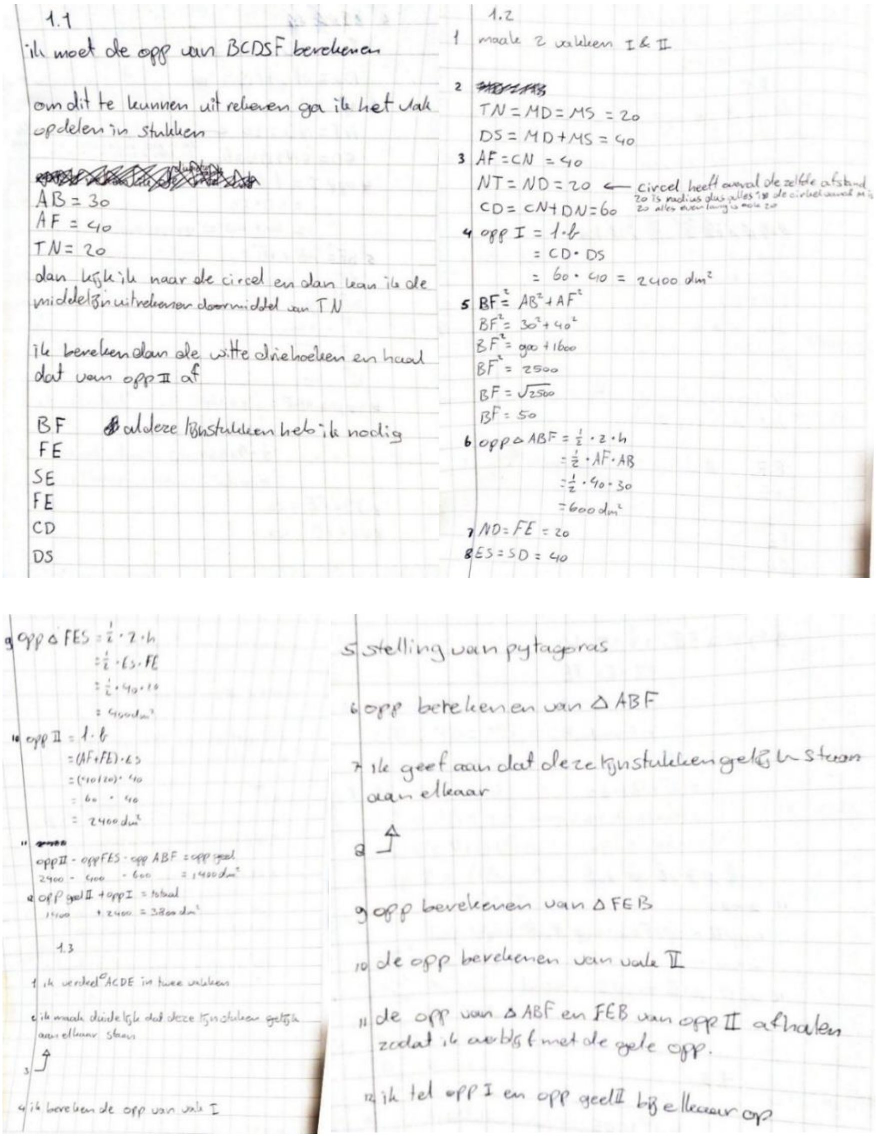


Note: As the assignment was handed in on paper, the solution is portrayed in the original Dutch. A translation is provided in the appendix.

On the other hand, students who struggled with mathematics or were insecure about their abilities and skills put a lot of effort into writing up their solutions. These students followed the method step by step and indicated feeling a sense of guidance when doing so. They used the method consistently throughout each problem set and were also seen utilising the method outside of the study to solve homework they had trouble with. Even within the classroom and towards their peers, these students had a very positive attitude towards the method, often claiming its merits for their process and showing growth in their ability to be able to discuss mathematics and solutions (core objective 19).

Figure 3

Solution to problem set 1 by an insecure student (research group).

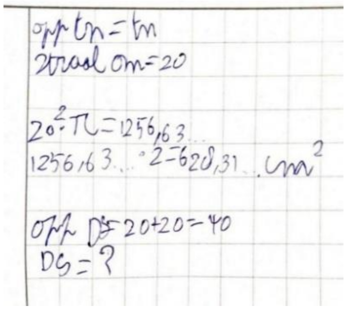


Note: As the assignment was handed in on paper, the solution is portrayed in the original Dutch. A translation is provided in the appendix.

While glaring differences in approach to the use of the problem-solving method appeared between students of the research group, just as glaring differences appeared immediately between the work of the students belonging to the research group and that of the students belonging to the control group. Within the control group, in fact, almost no student provided a sketch of the situation or an explanation of the calculations being made when working through problem set 1. An extreme example of this may be seen in Figure 4.

Figure 4

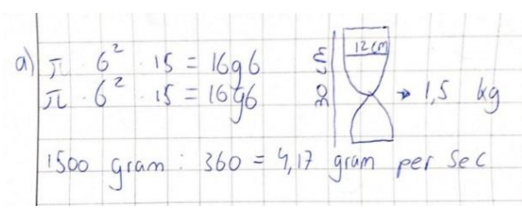
Solution to problem set 1 by a control group student



The differences between the two groups became even clearer later on, especially when looking at the number of students being able to correctly solve the given problem sets. While almost all students in both groups were able to solve the assignment in problem set 1, the number of correct solutions increased across problem sets 2 and 3 for the research group, while it decline for the control group. This would seem to indicate a positive effect of the problem-solving method on the analytical thinking of students in the research group. To be noted is that, independently of the problem-solving method, almost all students from the control group also made sketches of the situation presented in problem set 3 without being asked to. However, this particular problem set did not contain itself a picture to begin with (unlike the previous two sets, which each contained a figure). The absence of a figure in this particular assignment might have contributed to this natural step in the development of core objective 27.

Figure 5

Solutions to problem set 3 by control group students



Inside phase

All students from the research group participated very actively and enthusiastically in the inside phase, showcasing a growth in how they discussed their solution with peers and how they gave feedback or asked critical questions about each other’s work (core objective 19). While initial nudges were necessary, particularly to the avail of making students also be able to evaluate the good aspects of their own work and that of others, working together and discussing about their solutions came very naturally to all of them. These sessions were taken very seriously by the students, who were very honest about their shortcomings with respect to the topics handled throughout the assignment and who could also come up with quite concrete (albeit not always very detailed) plans for how to improve their understanding. These plans were always followed through as evidenced by follow-ups with the students. The reflection sheets used during the inside phase showcased how used students are to evaluating their mistakes rather than the general quality of their work. Many of them had great struggles with being able to reflect on what had gone well in their process, particularly if the student had not managed to find a solution.

Interviews

During the interviews, students from both the research and control group were asked to first solve a problem while thinking out loud. This highlighted differences between the two groups. The problem in question had been designed so that students would struggle to solve it and might ask the researcher for clarifications. This was indeed the case, although students from either group requested aid in very different ways. Students from the control group asked very general questions with the goal of having the researcher give away how to reach the answer.

> **A**: Hmm…I don't get it.
> **Researcher**: What don't you get?
> **A**: How do you do this?
> **Researcher**: Look at the information given. Are you missing something?
> **A**: I don't know how to start. What's the first step?
> **Researcher**: What are you trying to calculate?
> **A**: How much the volume has decreased.
> **Researcher**: Indeed. How could you do this?
> **A**: But I have no numbers. How do you do this?

On the other hand, students from the research group approached their struggles in a different way. These students automatically started by sketching the situation (core objective 27), even though they were not asked to do as much, and, if stuck, could indicate what pieces of information they were missing to continue and could express their thought process (core objective 19).

> **B**: […] I have no information, right?
> **Researcher**: What does the problem say?
> **B**: I have these percentages, yes. But usually you apply those to a number. […] This is the formula for the volume, but I cannot use it.
> **Researcher**: Why can you not use it?
> **B**: Well, I don't know these, right? […] Am I allowed to fill in the percentages?
> **Researcher**: How would you do that?
> **B**: Oh no, wait…does it matter what r is?

Their questions were aimed at understanding how the given pieces of information could be used to continue. Students from the research group were more aware of their own knowledge and of the struggles encountered while solving the exercise. Unprompted, they also utilised aspects of the problem-solving method as a fall back to make the point of the situation. Additionally, students from the research group were found think of the three problem sheets as much more useful than students from the control group.

The interviews revealed that the research group quite liked the problem-solving method, but thought it required too much writing. Especially insecure students saw the method as crutches to lean on when stuck and had started using parts of it while doing their homework. They were enthusiastic about the inside phase. Students mentioned that working in groups with different people every time helped them realise that they could rely on one another to resolve doubts about their mathematical knowledge. This was also reflected in a shift in atmosphere during normal lessons as students would look to peers for guidance instead of immediately asking their teacher. Students felt like the inside phase had posed them as the experts, giving them confidence in their mathematical abilities and skills.

Chapter 8 test

Although this research is of a qualitative nature, the grades of the students from both groups are reported in Table 1.

Table 1

Grades obtained by both K–8 groups on the chapter 8 test

	Research group	Control group
Student 1	9.0	8.6
Student 2	8.6	8.4
Student 3	8.3	8.3
Student 4	8.1	8.1
Student 5	8.1	7.6
Student 6	7.8	7.2
Student 7	7.3	7.2
Student 8	7.0	7.1
Student 9	6.5	6.5
Student 10	6.2	6.2
Student 11	6.2	6.0
Student 12	6.2	5.8
Student 13	6.2	5.5
Student 14	5.9	5.5
Student 15	5.8	5.5
Student 16	5.6	5.3
Student 17	5.5	5.3
Student 18	5.1	4.7
Student 19		4.6
Student 20		3.8
Average	6.9	6.4

Note: In the Dutch school system, a grade is considered passing if its value is 5.5 or higher.

These grades does not tell the complete story, but it is interesting to remark that only one student within the research group did not pass, while five students from the control group obtained an insufficient grade. Of these five, two grades would be considered badly and one even gravely insufficient. In general, the research group obtained better grades than the control group and produced better tests, although the grades themselves do not point to a statistically significant difference. Students from the research group produced more text, indicating what they were trying to achieve and why and explaining the purpose of their calculations. The full sentences required by the problem-solving method were not always present, but their thought process was visible on paper. The control group students, on the other hand, barely produced any written text and often combined multiple mathematical steps into one.

Discussions

An advantage of conducting this study in the lower grades of a secondary school in the Netherlands is that more hours of mathematics are taught per week and, therefore, hours are available that can be used for the complete methodology involving both outside and inside phase. However, this takes too much time: the inside phase in particular requires time and care that cannot always be spent on it. In preparation for this study it was ensured that the previous chapter would be concluded earlier than originally planned so that there would be enough time for students to both participate in the study and work properly through chapter 8. This, of course, makes the possibility of standard use of this methodology in the arithmetic-mathematics curriculum doubtful. Even more problematic would be the standard introduction of the method in the later years of secondary school, which only has three hours of mathematics per week, and which is already very dense when it comes to content and material. These time issues could be mitigated by utilising the methodology less often during each handled chapter. In particular, the method could be introduced from the very beginning of secondary school to ease its standard use. The method should then be used on a weekly basis or multiple times a chapters in earlier years. The frequency of use should then be diminished to once or twice per chapter with more exercises per sheet or higher level exercises per sheet leaning on the fact that students will in later years be used to the method and need not be taught how to reflect upon their work.

Although all respondents eventually cooperated reasonably well, it was initially difficult to convince them of the usefulness of the problem-solving method, which was met with considerable resistance due to the amount of writing required. The problem-solving method was introduced through the final chapter of the material for K–8 classes, with the result that the students of the research group had had two years to get used to a certain way of working with their teacher and, therefore, already had started to form an image of mathematics as a subject in which not a lot of writing is necessary (Huang & Normandia, 2009). An even younger class would perhaps resolve this issue as it could be led through the subject under the assumption that mathematics is essentially a language and requires a lot of writing. Since not all research group students applied the problem-solving method as intended, only a limited picture of the benefits of the method across the entire class can be described: it could still be that the method is very useful, even for more confident students, but the little practice with regular assignments and the minimal amount of effort with and for the method did not provide any visible benefits and developments for this group of respondents (Yuan, 2013). Moreover, the intrinsic motivation of this group students towards the method was lacking and the extrinsic motivation was also not present due to the fact that no grade was linked to the assignments, which were therefore seen as extra work without reward. This is of course a shame, because the usefulness of the method should be regarded as separate from immediately visible numerical benefits for the students. Unfortunately, it has become quite common for students in the Netherlands to learn for grades (Inspectie van het Onderwijs, 2022; Inspectie van het Onderwijs, 2023; Inspectie van het Onderwijs, 2024) and future research could examine whether associating a grade with the method has positive or negative effects on its benefits. It is, however, likely that including grades in this process will either not impact or negatively impact the effectiveness of the method as fidelity in the completion of the problem sets may decrease (Mello, 2023; Shepard & Law, 2015). Clearly outlined by this research remain still the benefits of the methodology through the inside phase, which most helped students develop analytical thinking through discussion and reasoning (Murphy et al., 2014) and which seemed to positively affect students' attitude towards mathematics (Yapatang & Polyiem, 2022). A further point of discussion is the influence of the two teachers. Although both communicated and aligned their pace in treating chapter 8, they had very different styles of teaching, with the teacher of the research group tending to introduce more opportunities for scaffolding and to support a more relational type of learning and understanding. This too might have ensured that the method took well to the research group, which might have been receptive to it from the beginning because of this.

Limitations

This research was subjected to a number of limitations due to its small scale and the time constraints it was faced with because of its timing and the choice of respondents. This study took place across 4 weeks, during which portfolios were built per student in the research group consisting of three problem sets and three reflection sheets. The limited acquisition of data does not clarify whether the introduction of the methodology described in this paper may have any long term lasting effects on the students of the research group. Some bias was introduced with respect to the interviews and the limited time of those involved also limited the coding of the qualitative data collected for this study. Furthermore, it may be the case that the methodology worked especially well for the particular class and cohort involved in this research. For this reasons, future research should delve into this topic on a much greater scale, testing the methodology across a variety of classes, grades, schools and countries. This would minimise confounding variables such as style of teaching and curriculum.

Core objective 23 rarely came into view in this study due to the chosen chapter not offering sufficient opportunities to explicitly prompt the students to showcase their insight into accuracy and order of magnitude. Future research should more carefully focus on this aspect of analytical thinking, choosing better suited mathematical topics pr designing exercises more clearly aimed at this. Future research should also monitor the grades of both the research and control group over a long period of time, which this study could not do.

Conclusion

During this research, the portfolios built up by the students in the research group provided some insight into their development of analytical thinking within chapter 8 thanks to the introduced methodology. As showcased by the problem sheets, most students from the research group showed improvement in their ability to describe their own thoughts and logic in full written sentences, further expanding upon those skills while collaborating with classmates during the inside phases of the study. Furthermore, these students used aspects of the problem-solving method to complete regular homework: struggling and insecure students had a fixed step-by-step plan at hand used to view, study and tackle difficult assignments. Because of this and the collaborative experience derived from the inside phase, students found it easier to talk about mathematics and discuss among themselves, resulting in a more pleasant atmosphere within the classroom and in a more positive attitude towards the mathematics and each other. During this research, the teacher became slightly less essential as students could discuss and work together before asking questions. This gave the teacher more time to emphasize certain parts of material and to pay attention to students who were experiencing greater difficulties.

From the results it can be concluded that the introduced methodology seemed to have positive effects for the mathematical and critical development of the students in the research group: the understanding and test results of struggling students, for example, improved, with the research group obtaining all passing grades except for one. Core objectives 19 and 27 were also quite clearly fostered. Particularly core objective 19 saw big improvements, resulting in better group functioning and communication within the classroom and mastery of mathematical language to articulate one's own ideas and logic with both fellow students and the teacher. The inside phase made this particularly clear, with students being able to discuss and evaluate different solution approaches without the intervention of the teacher during the last inside phase. The ability to reflect and be critical of their own answers and work came into view in the form of awareness of their own knowledge, which enabled several students to subsequently determine the correctness of their own answer. This awareness became particularly clear during interviews, where the research group could pinpoint exactly what difficulties they were experiencing with finding the solution to the problem given to them, unlike the control group who could not indicate which pieces of information they were missing. Particularly thanks to the reflection sheet, students also developed a sense for both the type of their mistake and the correct steps in their process. That recognition of their own abilities additionally helped improve the students' attitude towards mathematics. Furthermore, visualizing data and outlining the situation became more and more natural for students throughout the problem sheets, showcasing a fostering of core objective 27. Core objective 23 remains not as clearly fostered, partly due to the fact that the mathematical topics underlying this research did not suit inquiry into it.

The proposed methodology fosters, therefore, students' analytical thinking to the extent that students are enabled to develop missing 'basic skills' and understand how to use these outside the problem-solving method. Using the problem-solving method, students started reading assignments from their homework more carefully and more often checked that they had understood the question properly before starting to solve it, thus promoting critical reading. Similarly, students improved their problem-solving skills by developing a deeper understanding of the material that allowed them to more easily make connections between parts of the chapter in later assignments within the methodology. Thus, while the studied effect remains limited by the constraints of time, timing and choice of respondents, visible differences emerged between the research and control groups and a visible development emerged in the students from the research group thanks to their portfolios, which showed a growth in produced explanations in mathematical language, understanding and approaching the question and visualization of data, all analytical thinking skills (Amer, 2005). Only core objective 23 remains unclearly developed.

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Appendices

A. Problem sets

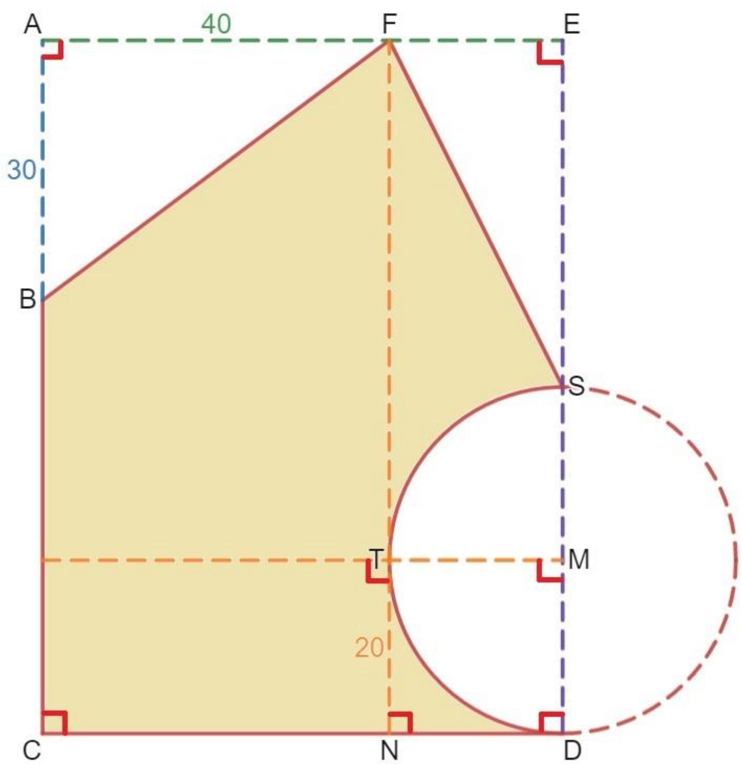
Here the assignments utilised during this research are included in translated form to English from the original Dutch.

Problem set 1

Calculate in dm^2 the area of *BCDSF* (the area coloured in yellow), where point *S* is the centre of the line segment *DE*, *M* is the centre of the line segment *DS* and the centre of the circle *c* and where line segment *AB=30cm*, line segment *AF=40cm* and line segment *TN=20cm*.
Round your answer to two decimal places.

Figure 6

Figure accompanying the assignment of problem set 1



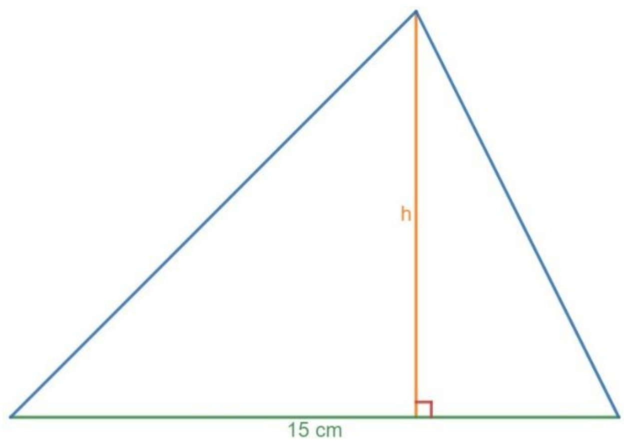
Note: The figure was created using GeoGebra and had no title in the original assignment.

Problem set 2

A flower shop wants to purchase new vases to replace the old ones. The old vases were shaped as a cylinder with a diameter of *15cm* and a height of *20cm*. The new vases have the shape of a prism, the base of which is shaped like a triangle with a side of length *15cm*, as can be seen in Figure 7. The new vases are just as high as the old vases.

Figure 7

The base of the new vases with a side of length 15 and line segment h



Note: The figure was created using GeoGebra. The title of this image has been translated from the original Dutch.

The flower shop wants to ensure that these new vases can hold more water and larger bunches of flowers. Determine what must be the minimum length in integers of the line segment *h* of the base of the new vases.

Problem set 3

To play a game, you would like to use an hourglass consisting of two equal-sized cones. The base of the cones has a diameter equal to *12cm* and the total height of the hourglass is equal to *30cm*. The sand in the hourglass fills an entire cone and weighs *1.5kg* per liter. After turning the hourglass over, it takes 6 minutes for the sand to flow into the other cone.

a) Calculate how many grams of sand flow through the opening of the hourglass per second. Round to two decimal places.

You want to build an hourglass yourself that is an enlargement of the hourglass you used for the game. With this new hourglass, the sand flows at the same speed as your answer from part a), but you now want to ensure that, after turning the hourglass, it takes 12 minutes for the sand to flow into the other cone.

b) What enlargement factor should you use to build this new hourglass? Round your answer to 2 decimal places.

B. Reflection sheet

Here the questions pertaining to third of the three reflection sheets are reported in translated form to English from the original Dutch. The sheet itself was designed using LaTeX and included spaces in between questions for students to write directly their answers. This appendix, however, serves as an example for the general structure of the reflection: the original layout is, hence, not included.

Reflection sheet 3

Name:

1. What tools did you use?
   Write down all the tools you used to solve the assignment, such as the internet, the book or your notes. If you asked teachers, classmates, or other people for help, specifically list everyone who helped you.
2. What went well?
   Look over your work after you have improved it. Which steps were successful?
   Now look specifically at step 2. Which calculations and steps were successful?
3. What type of mistakes did you make?
   Write down whether the errors you made were mainly comprehension errors (for example, you calculated something differently than asked or used the wrong formula) or calculation errors (for example, something that should have been a minus was calculated as a plus).
   Be careful: not knowing how to start or not arriving at the solution are not mistakes and do not need to be written down here.
4. What have you learnt from this session?
   Could you now complete similar assignments (properly)? Please explain why. Is this due to having seen the effect or having heard the explanation from a classmate?
5. Indicate per topic based on the discussion with your group, your work and the homework whether it is going well, requires repetition or requires to be looked at again and start over. Also briefly explain why it is going well or not.
   a. Volume of a pyramid
   b. Volume of a cone
   c. Determining enlargement factors
   d. From enlargement factors to surfaces
   e. From surfaces to enlargement factors
   f. From enlargement factors to volumes
   g. From volumes to enlargement factors
6. Write down a plan with which you will tackle the topics that you do not yet (entirely) understand. Be specific, so do not write ‘I am going to read the theory again’, but write for example instead ‘I still struggle with topics ... and … because ... and to improve that I am going to redo and self-assess assignments ... and … from pages... and …’.

C. Interview problem

After a candle, which has the shape of a cone, has burned for a while, the height has decreased by 32% and the diameter has decreased by 19%.

By what percentage has the volume of the candle decreased?

D. Translated figures

Below the translated text of various figures is translated.

Figure 2

dm^2 calculate surface BCDSF

surface square = side x side

Figure 3

1.1

I must calculate the surface of BCDSF

To calculate this, I will divide the surface in pieces

Then I take a look at the circle and then I can calculate the centreline by means of TN

I calculate then the white triangle and remove that from surface II

Other segments that I need

1.2

1. Create two surfaces I & II

3. Circle has everywhere the same distance. 20 is the radius, so everything on the circle is 20 away from M. Everything of equal length is also 20

11. surface II – surface FES – surface ABF = surface yellow

12. surface yellow II + surface I = total

1.3

1. I divide ACDE in two pieces

2. I make clear that these segments are equal to one another in length

4. I calculate the surface of surface I

5. Pythagoras theorem

6. Calculate the surface of triangle ABF

7. I indicate that these segments are equal to one another in length

9. Calculate the surface of triangle FEB

10. Calculate the surface of surface II

11. Remove the surfaces triangles ABF and FEB from surface II so that I am left over with the yellow surface

12. I sum together surface I and surface yellow II

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