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Eighth-Grade Students' Conceptions of Substitution: A Phenomenography

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Abstract: This study explores the conceptions of substitution held by eighth-grade students in the context of everyday life, using a phenomenographic research approach. Substitution, a critical concept in algebra, plays a central role in students' development of mathematical reasoning. However, despite its importance, students often struggle to apply substitution effectively in the classroom and real-world scenarios. Data were collected from 34 students across two eighth-grade mathematics classes using written responses and drawings. The analysis identified five distinct categories of students' understanding of substitution, ranging from basic perceptions of player removal and replacement in a sports context to more complex interpretations involving strategic decision-making. Additionally, the study revealed two qualitatively different views regarding the impact of substitution on team dynamics, with students perceiving either no change in the number of players or a decrease. These findings highlight significant variation in students' relational understanding of substitution and suggest that instructional approaches integrating real-world contexts may better support conceptual development. Teachers can foster deeper engagement and improve algebraic understanding by connecting students' informal experiences with formal mathematical instruction. The implications for teaching underscore the need for instructional designs that prioritize both conceptual and contextual learning.

Keywords: Substitution; Algebra; Student Conceptions; Middle School; Phenomenography.
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Introduction

The term substitution carries multiple meanings depending on the context, and these differences are not always evident. In mathematics, substitution refers to the formal act of replacing a variable or expression with an equivalent value, often as part of solving equations or simplifying expressions (Ayalon & Even, 2015; Jupri et al., 2016). The concept of substitution holds significant importance within mathematical education, serving as a foundational element in students' development of algebraic thinking. Scholars in mathematics education have underscored the pivotal role of algebraic substitution in enhancing students' understanding of mathematical concepts (Boero, 2002; Jupri et al., 2016; Kieran, 2007). Fey and Good (1985) further argue that the teaching of algebra should commence with inquiries into various aspects of substitution, including finding function values, solving equations, identifying extrema, determining rates of change, and calculating average values. Additionally, researchers have emphasized the utility of substitution in algebraic problem-solving, highlighting its role in elucidating underlying patterns and generalizations (Ayalon & Even, 2015; Jupri et al., 2016).

Studies in mathematics education have delineated two types of substitution in school algebra: replacing complex expressions with single variables and replacing single variables with complex expressions (Jupri et al., 2016). Furthermore, the concept of substitution transcends the pre-tertiary level. It pervades various topics in college mathematics, notably in Differential and Integral Calculus, where substitution facilitates the transformation of indefinite functions.

Beyond the confines of the classroom, substitution finds resonance in real-world contexts. For example, in everyday contexts such as sports, substitution may involve replacing a player due to fatigue or strategic considerations, an action embedded with notions of team roles, timing, and continuity (Martin & Gourley-Delaney, 2014; Nasir, 2000). These varied uses suggest that substitution is a technical and metaphorical concept, shaped by situational norms and purposes.

While mathematical substitution emphasizes formal equivalence and consistency, informal uses of the term often involve relational, contextual, or even emotional dimensions. Theoretical work on conceptual change highlights the challenges students face in connecting these everyday interpretations to formal mathematical contexts (Chi & Roscoe, 2002; Sfard, 2008). Moreover, science and mathematics education research points to the value of leveraging students' lived experiences to support deeper conceptual learning (Boaler, 1993; Ebenezer & Gaskell, 1995). In this study, we adopt a broad conceptual framing of substitution that includes its algebraic, contextual, and relational dimensions, setting the stage for a richer discussion of how this term functions across domains.

Scholars contend that informal experiences, such as participating in or observing sports events, provide valuable opportunities for students to engage with mathematical concepts in authentic settings (Benigno, 2012; Martin & Gourley-Delaney, 2014; Nasir, 2000; Ramini et al., 2015). Martin and Courley-Delaney (2014) advocate bridging the gap between school mathematics and everyday life, asserting that integrating these domains enhances the effectiveness of mathematical instruction and fosters students' mathematical proficiency.

However, despite its ubiquity and pedagogical significance, research has identified students' struggles with applying substitution in mathematical problem-solving (Filloy et al., 2010; Nogueira de Lima & Tall, 2008; Tirpáková et al., 2023). In response, recent studies have proposed pedagogical approaches to enhance students' understanding of substitution. Tirpáková et al. (2023) advocate for a three-stage instructional framework centered on developing students' conception of equality, understanding the structure of mathematical expressions, and honing transformational skills. Nonetheless, existing pedagogical approaches often overlook the importance of centering instruction around students' lived experiences, thereby neglecting the relational aspect of learning (Ebenezer & Gaskell, 1995).

To address this gap, the current study adopts a phenomenographic approach augmented with a photo-elicitation technique to explore eighth-grade students' conceptions of substitution in real-life contexts. Phenomenography, as described by Marton (1986), offers a methodological framework for documenting the qualitatively different ways in which individuals conceptualize phenomena. This study contributes to a growing body of research on informal to formal learning pathways(Boaler, 1993; Nasir & Hand, 2008) by investigating how students' intuitive reasoning in a familiar domain, such as sports, can anchor mathematical thinking. Although prior work has explored sports contexts for mathematics instruction (e.g., Dixon, 2008), the novelty of our contribution lies in how we positioned students as spectator analysts. This role invites students to mathematicise qualitative photo observations by reflecting on their experiences. Unlike traditional word problems or sports statistics worksheets, our approach foregrounds photo-elicitation as a dynamic medium for student inquiry, opening space for critical reasoning(Lesh & Lehrer, 2003). By examining students' written explanations and drawings, the study aims to elucidate the diverse interpretations of substitution and inform instructional practices rooted in students' lived experiences. The study seeks to provide an answer to the following question:

How do eighth-grade students express their relational conception of substitution through written explanations and drawings?

Studies on Substitution

Despite the ubiquity of the concept of substitution in mathematics education and its relevance beyond academic settings, a paucity of research is dedicated explicitly to exploring students' conceptions of substitution both within and outside the classroom. To address this gap, a comprehensive electronic search of scholarly literature was conducted across five databases: Google Scholar, Summon, PsycINFO, ERIC, and Web of Science. The search strategy combined terms related to substitution, student conceptions, and mathematics education. However, the search yielded no studies focusing on students' conceptions of substitution in educational and non-educational contexts. Nonetheless, a few related studies emerged that utilized the concept of substitution to understand other mathematical concepts, such as the equal sign and students' understanding of equations.

Nogueira De Lima and Tall (2008) examined high school students' conceptions of linear equations. Their findings revealed that when tasked with investigating equations such as 2m = 4m, students predominantly relied on memorized rules rather than understanding the concept of substitution. Similarly, Jupri et al. (2016) investigated the impact of digital technology on algebra learning, specifically focusing on students' approaches to algebraic substitution using an online Cover-up applet. Results indicated that the feedback provided by the applet facilitated error correction and strategy improvement among students, particularly in the context of word problems.

In a cross-national study, Jones et al. (2012) explored English and Chinese students' interpretations of the equal sign, including substitutive definitions. They argued that the substitutive definition of the equal sign is rooted in the mathematical principle of equivalence, emphasizing the relationship between substitution and equivalence. Building on this work, Simsek et al. (2019) provided further evidence of the predictive power of students' endorsement of the substitutive definition of the equal sign on their algebra performance. Their study demonstrated that students who endorsed the substitutive definition exhibited better academic performance in algebra, independent of other factors.

Furthermore, Donovan et al. (2022) investigated the effects of instruction on different conceptualizations of the equal sign on fourth and fifth-grade students' algebraic performance. Their intervention promoted a dual conception of the equal sign, incorporating both sameness and substitutive views. Results indicated that this instructional approach enhanced student abilities to produce sameness and substitutive definitions of the equal sign, highlighting the efficacy of addressing substitution in algebra instruction.

While research addressing students' conceptions of substitution remains limited, existing studies offer valuable insights into its implications for mathematics education. These studies underscore the importance of considering substitution as a foundational concept with a significant impact for students' understanding of algebraic concepts and mathematical problem-solving skills.

Mathematics outside of the classroom context

Numerous scholarly inquiries have emphasized the necessity of aligning mathematics instruction in educational settings with students' informal experiences (Cahyono et al., 2018; Esmonde et al., 2013; Martin & Gourley-Delaney, 2014; Scott et al., 2017; Voutsina & Scott, 2023; Wijers et al., 2010). In a recent investigation by Voutsina and Scott (2023), thirty-seven preschool children's perceptions of written numerical symbols within everyday contexts were explored. This study was conducted in the South of England and engaged preschool-aged children and their families in a "Number Spotting" game integrated into their daily routines. The findings of this inquiry served a vital educational function by furnishing novel insights into the diverse patterns of awareness that children cultivate concerning the utilization of written numerals in their surroundings. Moreover, the study unveiled the critical facets and foundational knowledge components that underline such variations.

Furthermore, Nasir (2000) documented a compelling instance where African American high school basketball players, through engaging in physical play, concurrently developed statistical understandings as they computed their individual and team statistics. This investigation underscores the intrinsic link between physical activities and mathematical cognition, illustrating how informal experiences can facilitate the acquisition and application of mathematical concepts outside traditional classroom settings. Children's mathematical experiences are deeply intertwined with social and cultural practices within their communities. Ethnographic studies have highlighted the cultural variability in children's mathematical activities, demonstrating how cultural norms and practices shape mathematical thinking and problem-solving strategies (Nasir, 2000). Children use mathematical reasoning through storytelling, games, and traditional rituals, reflecting their cultural values and social identities (Bishop, 1994). Furthermore, collaborative problem-solving and peer interactions are crucial in children's mathematical development within social contexts (Lave & Wenger, 1991).

Theoretical and Methodological Framework

Phenomenography has emerged as a prevalent theoretical, methodological, and analytical framework utilized by numerous scholars across various disciplines (Boda, 2021; Ebenezer et al., 2022; Hathway & Fletcher, 2018; Herbert et al., 2015; Voutsina & Scott, 2023). Originating from empirical inquiries conducted by Marton and his colleagues at Gothenburg University in Sweden (Marton, 1981), phenomenography offers a research methodology that underscores the multifaceted nature of the world and its susceptibility to diverse interpretations (Marton & Booth, 1997). Although phenomenography has been widely used in higher education (Akerlind, 2005; Prosser & Trigwell, 1999), its application in K–12 mathematics education, particularly in topics like algebraic substitution, remains relatively underexplored. This study responds to that gap by leveraging phenomenography to investigate students' conceptions of algebraic substitution, a core topic in algebra, foundational for understanding symbolic manipulation and problem-solving.

In STEM education, phenomenography has provided valuable insights into learners' conceptual variation and awareness. For instance, Scott and Voutsina (2024) applied phenomenographic analysis to understand students' conceptions of mathematical relationships in early arithmetic, highlighting how variation theory, closely linked to phenomenography, can inform instructional design. Similarly, Pettersson (2012) employed phenomenography to examine a longitudinal case study of one teacher candidate's understanding of the threshold concept of functions, demonstrating its utility in uncovering varying levels of abstraction and understanding. Phenomenography has been particularly useful in understanding how learners approach complex or abstract STEM concepts. For example, Linder and Fraser(2009) reviewed phenomenographic studies in science and engineering education, noting how the method helped uncover learners' differing conceptualizations of core principles like force and energy. In architecture education, Ebenezer et al. (2022) used phenomenography to examine architecture teacher candidates' conceptions of learning technology and found that most students perceive educational games as the most helpful teaching tools to consider in their future classrooms.

In mathematics education, the few studies have underscored phenomenography's strength in identifying variation in conceptual understanding, a key need in algebra instruction, where students often hold different ideas about variables and substitution (Kieran, 2006).

Moreover, Marton and Booth (1997) emphasize that phenomenography does not aim to describe the individual learner per se but rather to characterize the range of possible conceptions held by a group in relation to a phenomenon. This orientation makes it particularly valuable in mathematics education, where traditional cognitive or procedural analyses may not fully capture the nuanced and situated ways students interpret concepts like substitution.

Considering this, the present study adopts phenomenography to explore middle school students' conceptions of algebraic substitution, aiming to reveal qualitatively different understandings that can inform targeted pedagogical interventions. This approach aligns with recent calls in mathematics education for methodologies that go beyond error analysis and instead focus on how students construct meaning (Liljedahl, 2020; Sfard, 2008). By documenting the variation in students' experiences and conceptions, this study contributes to the development of instructional strategies that are sensitive to students' ways of understanding and the challenges they encounter when learning algebra**.**

Research Design and Method

This study represents a component of the culminating project undertaken by the first author as part of a doctoral seminar in STEM education during the winter semester of 2022 at a research-intensive institution in a Midwestern state. As part of the course requirements, graduate students were expected to develop final projects addressing local stakeholders' STEM education needs. Informed by the researcher's prior experience as a mathematics educator and a growing scholarly interest in pedagogically robust approaches to teaching and learning, the researcher initiated a collaborative inquiry with a middle school mathematics teacher. Approval was secured before the commencement of the study.

The research site was a public middle school enrolling approximately 629 students in grades six through eight. The student population is predominantly Hispanic, comprising 95.2% of the student body. African American and White students constitute approximately 2.5% and 1.7%, respectively, with the remaining percentage identifying as multiracial. A significant proportion of the students come from economically disadvantaged backgrounds, with many qualifying for free or reduced-price lunch programs.

Data collection occurred in Mr. Peter’s (pseudonym) two eighth-grade mathematics classes, consisting of 18 and 16 students, respectively. The gender distribution across the classes was relatively balanced, with 47% identifying as female and 53% as male. Mr. Peter, an early-career teacher in his early thirties, had five years of teaching experience. The professional development (PD) intervention focused on the curriculum unit of algebraic substitution, which aligned with the instructional goals of the teacher’s ongoing unit.

The primary aim of the collaboration was to engage the participating teacher in a professional learning experience that centered on eliciting and categorizing students’ mathematical thinking through the integration of a real-world phenomenon. The PD sessions totaled five hours and were delivered over three consecutive days via the Zoom platform. During these sessions, the researcher and the classroom teacher co-designed a lesson centered on algebraic substitution, integrating curriculum objectives and pedagogical strategies to support student sense-making.

Data collection and analysis

The instructional session commenced with an exploratory activity facilitated by the teacher, who distributed a worksheet featuring an illustrative depiction of a basketball match between two teams, namely the Lakers and the Bulls. A basketball match was used as an exploration activity because it emerged as one of the top three extracurricular preferences for the students in the two classes. This activity aimed at probing students' conceptualizations of substitution within the context of a basketball game. Employing the Prediction-Explanation-Observation Model developed by White and Gunstone (2014), students were tasked with providing written and visual responses to two second-order inquiries: (I) If you were the coach of the Lakers, how would you execute a player substitution? (II) Following the substitution of a player from your team, what do you anticipate happening to the total number of players?

The analytical process followed the established phenomenographic criteria for developing descriptive categories: the categories must represent distinctive ways of experiencing the phenomenon; a logical structure or hierarchy must exist among them; and the number of categories must be definite yet sufficient to represent the variation in the outcome space (Marton & Booth, 1997). Data analysis began with a period of immersion and familiarization, during which responses were read in their entirety without annotations to gain an intuitive understanding of students' perspectives.
The second stage involved coding and highlighting key phrases and images that served as indicators of meaning. The third phase involved grouping responses based on relational similarities and clustering them into preliminary descriptive categories that reflected critical and distinctive ways students conceptualized substitution.

Although some readers may associate phenomenographic research with the philosophical underpinnings of phenomenology, particularly the notion of "bracketing". This study’s methodological stance adhered to the applied interpretive aims of phenomenography. The term "bracketing" here refers not to the transcendental reduction of Husserlian phenomenology, but rather to an intentional effort to set aside researchers’ preconceived notions during the coding and categorization process (Ashworth & Lucas, 2000). In line with this goal, the researchers intentionally deferred engagement with a comprehensive literature review on substitution until after the core categories were developed. This approach was taken to ensure that the analysis was grounded in the empirical data and not unduly influenced by established theories, echoing arguments by Sandberg (1997) that premature engagement with theoretical constructs can obscure participants’ lived understandings.

That said, the decision to delay an exhaustive literature review does not imply the absence of scholarly engagement. Rather, it reflects a methodological alignment with phenomenographic goals: the descriptive categories are derived from participants’ expressed understandings rather than fitted into preexisting theoretical molds.
The final stage of analysis involved refining and naming the categories and organizing them into a structured outcome space. This involved articulating how different understandings of substitution were nested within one another and exploring whether some conceptualizations were more inclusive or sophisticated than others.

Figure 1

Exploration activity

EXPLORATION ACTIVITY

This is a basketball match between the Lakers and the Bulls

(1a). Imagine you are the coach of the Lakers. How do you think you will substitute a player from your team?

Explain your ideas

Draw your ideas

(1b). After substituting a player from your team, what do you think happened to the total number of players in your team?

Explain your ideas

Draw your ideas

Results and Interpretation

Tables 1 and 2 present the descriptive categories derived from students' written responses and drawings elicited in response to the two second-order questions. Table 1 delineates five distinct, descriptive categories formulated in response to the second-order inquiries. These categories encompass taking a player out, substituting one player for another, a player assuming the position of another, providing advice to the team, and adding a player. Additionally, two categories emerged concerning students' conceptualizations of the remaining players after a substitution event.

The presentation of results follows a structured approach: (1) The introduction of the table, which outlines the descriptive categories based on the second-order questions, a representative example extracted from students' responses, and the corresponding frequency distribution. (2) After the tabular representation, a descriptive commentary is provided, which compares students' responses and elucidates the mathematical intricacies embedded within them. (3) Finally, the findings are contextualized within the existing literature, establishing connections and drawing parallels to relevant scholarly discourse.

Table 1

Descriptive categories, students' conceptions of substitution, and frequency

Descriptive Categories	An Example of Students' Response	Frequency (n=34) Writing	Drawing
F. Advising the team	One or two days before the game I will call all of the players and tell them of the plan that can work on the game	2	2
G. Adding a player	I would substitute a player by adding another player that isn't playing	1	1
H. Taking a player out	To substitute another player I would take a player out that seems tired.	12	12
I. Changing a player with another player	I will have to take him out and put someone on the bench in the game	13	13
J. Having a player from the bench take the position of a player	To substitute a player from my team, I think I would have another player take their position	6	6

Descriptive Category F: Advising the team

Table 1, two (5.9 %) students believe substitution means offering advice to their team. Note excerpts below.

Figure 2

Mike's drawing

One or two days before the game I will call all of the players and tell them of the plan that can work on the game (Mike)

Figure 3

Tara’s drawing

I would tell my team to get their stuff and tell them to go out and play like it's the last game think about your family and the most important things. Have fun (Tara)

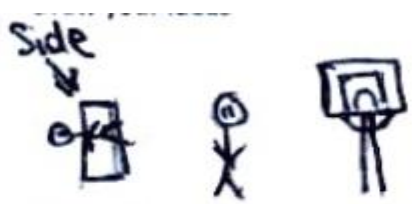
Mike and Tara interpreted substitution within the context of guiding a group of athletes. Tara perceives this guidance as a source of motivation for the players. At the same time, Mike views it as a strategic plan for securing victory. Their respective drawings depict varied aspects of the substitution phenomenon, illustrating a coach offering advice and encouragement to the team. This observation aligns with Marton and Tsui's (2004) assertion that individuals may approach problem-solving differently based on their unique perspectives. Marton and Tsui emphasize that varying interpretations of a problem can influence individuals' problem-solving abilities, reflecting the diverse cognitive processes at play during problem-solving tasks.

Descriptive Category G: Adding a player

One (2.9%) student viewed substitution as merely adding a player to the team. Observe what Eva had to say:

Figure 4

Eva's drawing



I would substitute a player by adding another player that isn't playing(Eva)

Eva strongly emphasizes recruiting a new player to join the team, with comparatively less attention given to replacing an existing player. In Eva's illustration, she directs attention to a player positioned on the sideline, awaiting entry into the active team lineup. However, her written response and drawing fail to highlight any indication of a player being substituted out of the game. This perspective underscores a fragmented comprehension of the substitution phenomenon, as Eva focuses on one aspect of the process

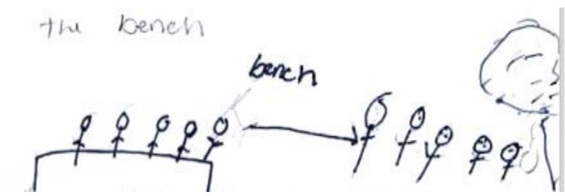
The distinction between categories G and F is notable in that students under G appear to address a component of the substitution process. In contrast, those in category F demonstrate a lack of consideration for any methods associated with substitution.

Descriptive Category H: Taking a player out

Twelve (35.3%) students responded directly opposite to category G. We give representative excerpts below:

Figure 5

Tom's drawing



Tell the player to sit down on the bench. (Tom)

Figure 6

Peter's drawing

I will take a guy out from the court. (Peter)

Figure 7

Liana's drawing

To substitute another player I would take a player out that seems tired. (Liana)

Figure 8

Paul's drawing

Well, I think if you need to take away a player like if they get hurt or like they need some time to relax or something. (Paul)

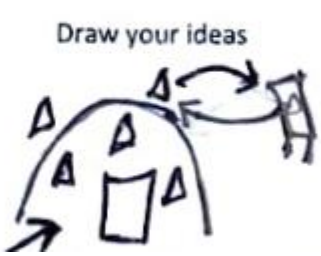
Across all responses, the overarching theme is that substitution entails the removal of a player from the team. Liana and Paul displayed discerning attitudes towards selecting the player to be substituted. Liana focused on identifying a fatigued player, whereas Paul focused on replacing an injured player. Conversely, Tom and Peter notably should have commented on substituting the specific player. Although Tom indicated a player transitioning towards the bench, he failed to provide explicit evidence of a replacement for the departing player. Notably, this subgroup of students predominantly construed substitution solely as removing a player, reflecting a fragmented comprehension of the concept. This narrow conceptualization underscores a limited understanding of substitution. The disparity between categories G and H illuminates contrasting manifestations of fragmented understanding, with some students demonstrating nuanced consideration of the substitution process. In contrast, others exhibit a more simplistic perspective.

Descriptive Category I*: *Changing a player with another player

Thirteen (38.2%) students in category J conceptions of substitution seem to encompass ideas from both categories H and I. Note the excerpts below:

Figure 9

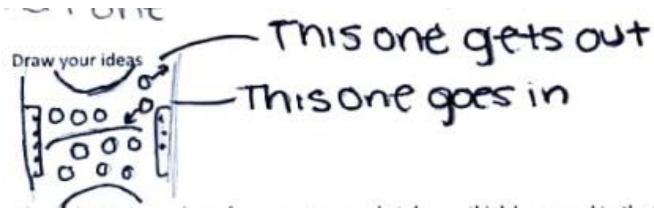
Ike's drawing



I will have to take him out and put someone on the bench in the game(Ike)

Figure 10

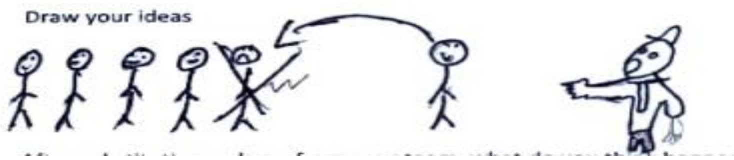
Lee's drawing



This one gets out –This one goes in. (Lee)

Figure 11

Abi's drawing



If you need to substitute a player you replace the injured player with for another player. (Abi)

Figure 12

Rab's drawing

I would say to stop the game and change the player that I want to change to the new one.(Rab)

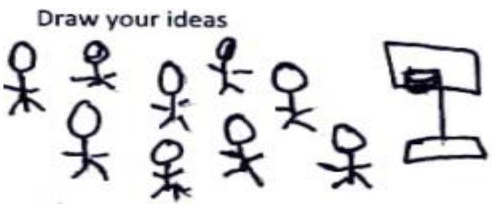
The written and visual responses from Ike and Lee reveal students' conceptualization of substitution as a dual process involving removing one player and adding another to the playing team. However, their contributions need more substantial evidence regarding whether the incoming player assumes the position of the outgoing player. In contrast, Abi and Rab offer additional insights by specifying which player should be replaced and outlining the procedure for executing the substitution. While one student addresses the issue of an injured player, the other emphasizes the necessity of halting the game before initiating the substitution process. Notably, responses categorized under J demonstrate a more coherent understanding of substitution than categories H and I.

Descriptive Category J*: *Having a player from the bench take the position of a player

Six (17.6%) students' responses extended the central idea in category J with an additional layer of information. We provide representative excerpts below:

Figure 13

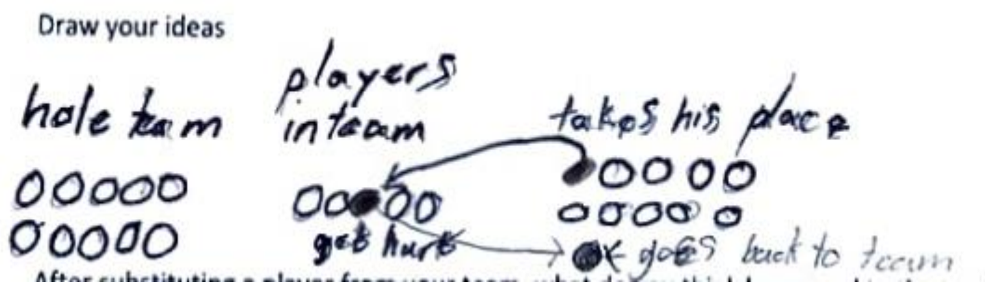
Joe's drawing



To substitute a player from my team, I think I would have another player take their position. (Joe)

Figure 14

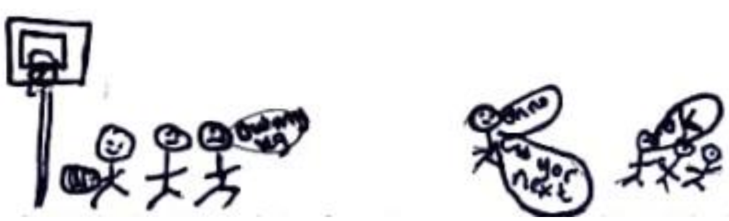
Sarah's drawing



You would take him out and put another player in to take his place. (Sarah)

Figure 15

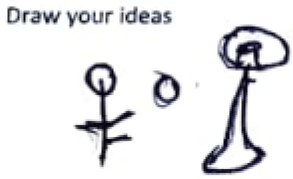
Rachael's drawing



would substitute a player if they got hurt. I will tell the next player that is ready to play to take the other's players place. (Racheal)

Figure 16

Nick's drawing



I would check the positions they play from the player that get out(Nick)

The excerpts not only delineate students' understanding of substitution as the removal of one player and their replacement with another but also underscore the importance of the incoming player assuming the precise position of the outgoing player. This descriptive category indicates a heightened conceptualization level, as Marton and Saljo (1976) proposed. Notably, Nick emphasizes the necessity of assessing the player's status to be substituted as the initial consideration in the substitution process. Meanwhile, Joe, Steve, and Rachel are concerned about the incoming player's alignment with the position vacated by the outgoing player. Rachel, in particular, provides a comprehensive plan outlining the requisite conditions for executing the substitution.

Category K further elucidates the evolution and extension of students' ideas across different descriptive categories, highlighting a hierarchical progression. The referential component of category K is particularly noteworthy as it illustrates the expansion of students' concepts from category H to K, thereby enhancing inclusivity within the descriptive framework.

Table 2 below indicates two qualitatively different ways students responded to the question:
After substitute

Table 2

Categories, an example of students' conceptions of what happened to the total number of players after substitution, and frequency

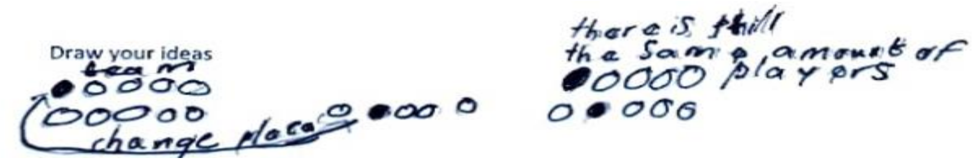
Descriptive Categories The number of players will…	An Example of Students' Response	Frequency (n=34) Writing	 Drawing
L. Remain the same	After substituting the number of players stay the same it would just be a replacement	24	21
M. Decrease	I would lose a team player and it would be harder to win with just four players	7	7

Descriptive Category L: ***The number of players will remain the same***

Twenty-four (70.6%) students believed the number of players would not change after substituting a player. Note the excerpts below.

Figure 17

Sarah's drawing



There will still be the same number of players in the team. (Sarah)

Figure 18

Ramz drawing

After substituting the number of players stay the same it would just be a replacement. (Ramz)

Figure 19

Ike's drawing

The total number of players don't change because we aren't subtracting, we are substituting. 5-1= 4, 4+1=5 (Ike)

Figure 20

Judy's drawing

I would take the one player out and put someone else in their position so the team will have the exact number of players(Judy)

Sarah accompanied her written explanation with a drawing depicting an arrow linking the departing player and another player poised to join the team. Similarly, Ramz and Judy utilized two opposing arrows to represent the player exiting the game and entering the lineup. In contrast, Ike employed two mathematical equations to substantiate his argument. Specifically, Ike assumed a total of five players, resulting in four after the departure of one player, as illustrated in the first equation. Subsequently, he accounted for the incoming player by adding one to the remaining four players in the second equation. Ike's detailed approach underscores a deeper level of conceptualization, aligning with the theoretical framework proposed by Marton and Saljo (1976).

Descriptive Category M: The number of players will decrease

Seven (20.5%) students believed the number of players would reduce after substituting a player in a basketball game. We provide representative excerpts below:

Figure 21

Ben's drawing

What will happen to the number is they will go down from 10 to 9 because that one player is out. (Ben)

Figure 22

Mike's drawing

The number of my team would decrease because one of my players got hurt, so I will have less players. (Mike)

I would lose a team player and it would be harder to win with just four players. (Pat)

I would be 1 less. (Dan)

Ben adopted a mathematical approach by assuming ten players and employing subtraction to illustrate that removing one player from the team results in nine remaining players. Mike corroborated Ben's assertion by noting a reduction in the number of players following the departure of an injured player. Although Pat and Dan did not accompany their responses with visual representations, their written explanations were clear and aligned with the insights provided by Ben and Mike.

By comparing categories L and M, a qualitative difference in structural terms becomes evident. One group demonstrated a more comprehensive understanding by conceptualizing substitution as a reciprocal process. Consequently, they concluded that the number of players remains unchanged after substitution. Conversely, the other group perceived substitution merely as removing a player without introducing a replacement, indicative of a superficial understanding of the concept.

Discussion

The findings presented here shed light on the students' diverse conceptualizations of the substitution phenomenon. The interpretations vary significantly, from viewing substitution as a motivational tactic to considering it solely as a tactical move aimed at winning. This divergence in perspectives indicates Marton and Tsui's argument that individuals may approach the same problem differently based on their unique perceptions and understandings.

The study identifies distinct categories of understanding, highlighting how students perceive substitution. For instance, some students focus on bringing a new player onto the field. In contrast, others concentrate solely on removing a player from the game. This discrepancy in emphasis suggests a fragmented understanding of the substitution process among certain groups of students.

Recent studies in mathematics education address the educational implications of conceptual variability in instruction. For example, research by Stein et al. (2008) emphasizes the importance of facilitating productive discussions and mathematical discourse in the classroom to promote students' conceptual understanding and sense-making. Elsewhere, Cho and Kim (2020) emphasize the influence of students' beliefs and conceptual understanding on problem-solving approaches in mathematics, suggesting that individual differences play a crucial role in how students interpret and approach concepts. This variability is evident in the diverse perspectives observed among the participants, ranging from viewing substitution as a motivational tactic to considering it solely as a tactical maneuver aimed at winning.
Moreover, the analysis reveals a spectrum of comprehension levels, ranging from surface conceptions to deeper conceptualizations. Students who consider substitution as a two-way process, ensuring that the incoming player assumes the exact position of the outgoing player, demonstrate a more profound understanding. This aligns with Marton and Saljo's framework, which emphasizes the depth of conceptualization in learning processes. Recent research by Kyriakides and Creemers (2020) highlights the importance of fostering deep conceptualization in mathematics education. Their meta-analysis of experimental research underscores the significance of instructional interventions that promote meaningful learning outcomes by nurturing students' conceptual understanding. In this study, the depth of conceptualization varied among students, with some demonstrating a more profound knowledge of substitution while others exhibited surface-level conceptions.

Additionally, the study identifies a hierarchical progression in students' ideas, with some categories building upon others, indicating an evolution and extension of understanding over time. This evolution reflects a dynamic learning process wherein students refine their conceptualizations through continued engagement and reflection. Pitta-Pantazi et al., 2015 offer insights into students' mathematical thinking progression, emphasizing the sequential development of procedural and conceptual knowledge. This framework can be applied to the findings of this study, where students' understanding of substitution progressed from surface to deep conceptualization. By recognizing this progression, educators can design learning experiences that scaffold students' understanding and promote meaningful engagement with the concept.

Conclusion

In conclusion, the study elucidates the multifaceted nature of students' conception of substitution in a sports context. Through qualitative analysis, distinct categories of understanding emerge, reflecting diverse perspectives and levels of comprehension among the participants. The findings underscore the significance of recognizing and addressing these variations in educational practice to promote deeper learning and understanding. By acknowledging the nuanced ways in which students perceive and interpret concepts, teachers can create more inclusive and effective learning environments that cater to the diverse needs of learners. Moreover, the study contributes to the broader discourse on conceptual learning, highlighting the importance of considering individual perspectives and conceptualizations in educational research and practice.

The phenomenography approach to exploring and categorizing students' conceptions was instrumental in eliciting students' ideas. Second-order questions created a space for all the students to share their ideas about the natural phenomenon without fear and intimidation. Using phenomenography categories, the teacher shifted the focus to the variations in students' ideas rather than categorizing students into high and low achievers. This action is an expression of intellectual empathy towards students' ideas. Here, the categories are not used as misconceptions or preconceptions but rather as variations. As a result, the teachers' focus was not on repairing learners' misconceptions (Chi & Roscoe,2002) but on providing them with experiences to explore the outcome space. Phenomenography offers an alternative way of determining students' competence which diverges from the traditional deficit way of examining students' multiple intelligence. (Gardner, 1993). The findings from the study illustrate qualitatively different ways the students understood the two simple exploration activities based on a natural phenomenon.

Several studies have emphasized the significance of a student-centered approach to teachingand learning mathematics (Haser & Behiye, 2003; Pedersen & Bjerre, 2021). Yet, there's still a lack of clarity on how to embark on this journey.Instead of solely focusing on procedural fluency, teachers should prioritize the development of conceptual understanding in mathematics. This entails providing opportunities for students to explore the underlying principles of substitution, make connections between mathematical concepts and real-life situations, and engage in meaningful problem-solving tasks. Emphasizing conceptual understanding helps students develop a deeper appreciation for mathematics and equips them with transferable problem-solving skills. From this current study, students' conception of the exploration activities would serve as the basis for lesson development and implementation in the subsequent phase of lesson development. This approach will help students monitor their conceptual growth. Students will see learning as a qualitative change from their perspectives to a deeper and more complex understanding within a specific context (Marton & Tsui, 2004). Teachers are therefore encouraged to adopt this approach to preparing their curriculum materials as it allows them to be sensitive to students' voices and helps to balance the power structure in the learning environment (Cohen & Lotan, 2014).

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