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Development of Number Sense and Numeration: A Continuum Hypothesis

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Abstract: This text focuses on the development of number sense, specifically addressing the challenges associated with understanding the number system, including the connections to and difficulties in early arithmetic learning. It provides a synthesis of the scientific literature related to this development. It concludes by proposing a hypothesis for the development of number sense, associated with the construction of dynamic and pictorial mental representations. This continuum proposes five key phases: perception of small quantities, additive thinking, multiplicative thinking and pre-place value, passage to tens and passage to hundreds and understanding place value.

Keywords: *mathematics; learning difficulty; number sense; numeration; subitizing; groupitizing; mental representation.*

Introduction

Mathematics are important in students' academic careers as it plays a crucial role in their educational progression. Success in elementary school mathematics enables students to continue their studies through the regular pathway in high school. Furthermore, proficiency in mathematics at the high school level opens doors to various academic pathways that may otherwise be inaccessible. These pathways can lead to college opportunities in fields such as science or management (Dionne, 2007; Geary, 2000). Number sense serves as a fundamental component of mathematics and is a significant predictor of success not only in mathematics but also in other subjects like reading (Case & Okamato, 1996; Claessens, Duncan & Egel, 2009; Duncan et al., 2007; Pagani et al., 2011). According to the literature consulted, no clear theory or hypothesis seems to define how number sense develops and in what conditions. Nonetheless, existing data suggest that a critical aspect of number sense development lies in the comprehension of numeration. Taking this into account, a better understanding of children's comprehension of numbers before numeration is sought, aiming to propose solutions that support students who encounter difficulties and potentially prevent their emergence. Consequently, this paper initially defines number sense as understood within the context of the study. Subsequently, an updated synthesis of the literature review done in the author’s thesis (Bisaillon, 2021), in relation to the development of number sense, is proposed. Following this, a hypothesis concerning the development of number sense, associated with the construction of dynamic and pictorial mental representations, is formulated. This hypothesis is based on the available literature. Finally, a brief conclusion is provided, outlining new avenues for future research.

Number sense: its characteristics ans its challenges

Sayers et al. (2016) differentiate three thought processes when dealing with number sense. The first one is innate or preverbal and is believed to emerge before the age of 4 (Starkey & Cooper, 1980; Wynn, 1992a; Wynn, 1992b). It is the ability to perceive, with precision, small numbers (from 0 to 4 elements) or to compare, in an approximate way, quantities in the order of single to double (Dehaene, 2001; Jordan & Levine, 2009). The second thought process associates number sense with grade one-level knowledge of numbers and their relationships. This comprehension requires instruction. Sayers et al. (2016) call this "Fundamental number sense" (FoNs). The researchers identified eight components of FoNs that are both distinct and related: number recognition, counting, awareness of the relationship between number and quantity, quantity discrimination, understanding different representations of numbers, estimation, arithmetic competence, and awareness of regularities in numbers (Sayers & Andrews 2015; Sayers et al., 2016). The third thought process was named "applied number sense" by Sayers et al. (2016) and refers to "the fundamental number-related understanding that infuses all mathematics learning" (Sayers et al., 2016, p. 473). According to this thought process, number sense is characterized by flexibility in number manipulation that reflects a true understanding that goes beyond the correct execution of techniques or routines (Bobis, 2007; Reys, 1994). This flexibility occurs in the ability to compare numbers and compose and decompose them (Brissiaud, 2005). Number sense evolves throughout life (Reys, 1994). The present study is part of this approach.

According to Reys and Yang (1998), number sense refers to an individual's general understanding of number and arithmetic operations. It is a way of thinking (Reys, 1994). The development of number sense is characterized primarily by the ability to understand number-based situations and the effects of manipulations on numbers. Number sense is a complex concept with many variables involved. Given this complexity, Reys (1994), based on his observations of students, attempted to establish a list of behaviors associated with number sense, the presence of which may vary according to the situations that are proposed to the students. This list includes making connections between mathematics and the real world, be able to judge the accuracy of an answer, invent their own calculation procedures, represent a number in several ways depending on the context, recognize patterns in the number system, easily obtain the answer to elementary calculations, be able to use their knowledge of numbers to deduce new ones, regularly use estimation, play with numbers by composing or decomposing them, recognize the size and quantity behind the numbers. These components are not exclusive to number sense, especially the first two. Some of these components are also like those proposed by Sayers and her colleagues for FoNs (2015, 2016). However, they all express this idea of flexibility, of understanding numbers, of their relationships and of the ability to manipulate them according to the tasks to be solved related to number sense.

Witzel and colleagues (2007) noted that students with weak number sense often struggle with perceiving the significance and function of grouping within our number system. Our number system contains both additive and multiplicative aspects. Each digit is multiplied according to its position and all these products must be added together to find the total quantity (4 x 100 + 5 x 10 + 6 x 1 = 456). Moreover, our system is made of groups and groups of groups of tens. It is based on equivalence rules (10 ones equal 1 ten). Finally, our system is positional; the digits have different values according to the position they occupy in the number. Therefore, behind the writing of numbers with digits are groups of tens, of hundreds, etc. In a landmark study in Quebec, twenty-five years earlier than Witzel and colleagues (2007), Bednarz and Janvier-Dufour (1984a, 1984b, 1986) had noticed that students had difficulties to see and understand the groupings implied by the positional writing, in other words, in conceptualizing them. They had difficulties to recognize the quantity behind the place value. Thirty years later, Koudogbo and colleagues (2013, 2017) assessed 18 third grade students (8-9 years old) using the same tasks used by Bednarz and Janvier-Dufour (1984a, 1984b). They compared their students' scores to those targeted in the 1980s studies. It appears that the performance of these students did not differ from the students targeted in the studies conducted in the 1980s (Koudogbo, 2013). The perception of the relevance of grouping in our number system seems to be a significant and persistent difficulty.

In addition, results from Bednarz and Janvier-Dufour (1984a, 1984b, 1986) show that students who receive more systematic instruction in certain number sense concepts, including perceived relevance of grouping, perform better on mathematical tasks related to numeration. Jordan and colleagues (Jordan, 2010; Jordan & Dyson, 2016) number sense studies support the idea that working on more primitive number sense concepts, such as subitizing, the ability to instantly perceive small quantities, may facilitate understanding the number system. Further research is needed to confirm these findings and to better document the place of more primitive number sense concepts, the concepts that underlie numeration learning. It is also important to clarify these concepts and determine their role in the development of number sense.

Many authors (Greeno, 1991; Howden, 1989; McIntosh & Dole, 2000; Reys, 1994) agree that number sense is something that develops throughout life. The present study focuses on this development up to the point of understanding our multi-digit system. However, some studies are interested in this development in young children and others are interested in the development related to the learning of numeration. Other studies, such as those by Clements & Sarama (2004, 2009), propose learning trajectories for certain elements associated with number sense, but do not suggest one for number sense per se. It seems, therefore, that further reflection is needed to identify, through the work available in the scientific literature, a continuum that would account for the development of number sense. The objective of the present paper is to identify, through the work available in the scientific literature, how number sense develops and to propose a hypothesis for its development.

Number sense development: overview of key studies

To document the development of number sense, a review of the literature was conducted. Initially, since this development is associated with the understanding of number, studies that specify what understanding place value system mean were identified. Next, the focus was on studies interested with the development of number sense. The studies selected attributed similar characteristics to number sense as those presented above. However, they proposed details linked to certain phases of this development, rather than detailing it in its entirety. Thus, the work of Jones et al. (1994, 1996) on the understanding of multi-digit number sense served as a starting point for this part of the literature review. This work led to studies of Clark and Kamii (1996) and Jacob and Willis (2003) on the acquisition of multiplicative thinking, an important phase mentioned by Jones et al. (1994, 1996). Then studies on the early manifestations of number sense presented in the previous models were consulted. The work of Brissaud (2005) on the early acquisition of number has been reviewed. His proposal regarding two forms of counting strategy (sequential counting and global counting) guided further research. In the present study, the focus was primarily on global counting, because it refers more to the understanding of numbers. To gain a better understanding of global counting, relevant research on groupitizing and subitizing (Baroody, 2004; Brissiaud, 2005; Clements, 1999; Clements et al., 2019 Dehaene, 2003; Houdé, 2004; Schindler et al., 2020, Starkey & Cooper, 1980; Starkey and McCandliss, 2014; Wynn, 1992a, 1992b) was consulted. To present the results of this literature review, what it meant by understanding our place value system is first exposed. Secondly, studies that focus on moments in the development of number sense will be discussed.

Understanding place value numeration

Understanding each of the features of the number system is a gradual process, the goal being helping the students develop flexibility in manipulating numbers when they are presented in their decimal positional form. Comprehension is the result of building connections between different representations of the same mathematical concept (Duval, 1996; Hiebert & Wearne, 1992). External representations, those perceived by the senses, can be concrete, pictorial, symbolic or verbal (Hierbert & Wearne, 1992). They play an important role in learning and constructing solid and coherent mental representations (Bednarz and Dufour-Janvier, 1984b, 1986; Thomas & Mulligan, 1995; Thomas et al., 2002).

Internal or mental representations, on the other hand, are those that the person mentally evokes. They are essential for the learning process (Racicot, 2008). These representations can be pictorial, symbolic, verbal, or emotional. Students need to use mental images of numbers and the relationships that numbers have to each other to eventually find their own way to solve mathematical tasks (Sullivan, 2018). Understanding the number system is therefore the ability to recognize relationships between different external representations and processing them to form an internal mental representation (Hiebert & Wearne, 1992).

According to Clements and Battista (1992), the construction of mental representations play an important role in mathematical thinking in primary school, but also in advanced mathematics. In addition, Thomas, Mulligan, and Goldin (2002) found that students who had a strong understanding of numeration showed evidence of dynamic internal pictorial representations. These students were able to represent our base ten number system mentally and could manipulate the elements, i.e. move, change, or transform them in their minds (Thomas & Mulligan, 1995). This pictorial and dynamic mental representation evolve as number sense develops (Thomas et al., 2002).

Thus, number sense seems to be related to the understanding of numbers and the relationships between them. This understanding seems to be associated with the construction of dynamic and pictorial mental representations of quantities. The construction of these representations also supports the development of flexibility in the manipulation of numbers. The next section reports on the studies consulted to document the development of number sense. A synthesis will then establish links between this development and the construction of dynamic and pictorial mental representations.

Studies on the development of number sense and numeration

A review of studies on number sense development was conducted. Some studies focus on the development of number sense in young children while others are concerned with learning numeration, but they do not necessarily combine these two aspects. The purpose of this review was to identify common and distinctive elements. In an effort to identify a possible cognitive pathway for a dynamic and pictorial representation of quantities, we decided to start with understanding numeration and gradually associate it with more primitive concepts of number sense, such as groupitizing and subitizing.The work of Jones et al. (1994, 1996) was used as a starting point for the literature review because it proposes a model for the development of multi-digit number sense, which is the culmination of the learning discussed in this study. They identified five levels leading to the student's ability to manipulate numbers and choose the best representation for the operation to be performed. These levels are described in Table 1.

Table 1

Description of the evolution of children's arithmetic thinking according to the work of Jones et al. (1994) (Bisaillon, 2021)

Levels	Description of the thinking
1 Pre-place value	• Thinking in terms of units only; • Difficulty decomposing and comparing multi-digit numbers; • No use of grouping.
2 Beginning of grouped counting	• Beginning to coordinate counting in terms of units and tens; • Beginning to understand grouping.
3 Using the ten as an abstract unit	• Coordination in counting in terms of units and tens; • Counting in steps of ten; • Flexibility in representing two-digit numbers (comparison, decomposition, operations).
4 Passage to hundreds	• Understanding of hundreds as a new unit; • Coordination in counting units, tens and hundreds; • Counting in steps of one hundred; • Flexibility in representing three-digit numbers (comparison, decomposition, operations).
5 Understanding of place value	• Flexibility in all components for three-digit numbers.

They created mathematical problems based on this categorization and presented these problems to students from grades 1 through 6. According to their results, the majority of grade 1 students, at the end of the year, are at least at Level 2. Half of the students in grade 2 are at level 3 or below at the end of the year, i.e., understanding ten as an abstract unit. The other half of the students are at level 5.

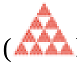
Since the present research focuses on the difficulties associated with understanding the relevance of the grouping principle occurring in the number system, Level 1 (pre-place value) and Level 2 (beginning of grouped counting) of Jones and al's (1994) model were studied in greater depth. Level 2 of Jones and his colleague (1994; 1996) refers to the onset of grouping. Grouping is the concept on which numeration is based (Fuson, 1990; Poirier, 2001). To be able to count a large quantity at a glance, it is necessary to form groups ( tens) and groups of groups ( hundreds). This makes it easier to access the cardinal (or number of elements) of the collection thus organized.

Figure 1

Example of visual and global organization of 324 elements (Bisaillon, 2021)

counting one by one — the organized collection

Despite its central importance, few students see the relevance of grouping and few students can explain how grouping helps to organize a collection, as previously mentioned (Bednarz & Janvier-Dufour, 1984a, 1984b, 1986). Students show that they understand the meaning of grouping when they can group or ungroup the various units used in the number system, when they can compose and decompose numbers and when they can manipulate them (Brissiaud, 2005).

Research by Clark and Kamii (1996) has shown that to understand the relevance and role of grouping in our number system, students need to have developed multiplicative thinking. This is not simply the ability to perform multiplication, but rather the ability to consider the relationship between the values to be multiplied and use multiplicative thinking in some way. Multiplicative thinking allows for the simultaneous processing of elementary units and higher order units such as units versus tens. Clark and Kamii (1996) propose levels of development related to grouping, one principle of the number system. To identify levels of development of multiplicative thinking, they presented a task involving multiplication to 336 students in grades one through five. They showed the children three fish of different sizes. The students were asked to give food to the fish, represented by counters, according to certain instructions. For example, they were asked how many counters would be given to the smaller fish if the larger fish received four counters. This research identified four levels of development from additive to multiplicative thinking, which may correspond to the first two levels of Jones and his colleagues (1994). Level 4 is associated with multiplicative thinking. For students who have not developed their multiplicative thinking, they refer to their thinking as simply additive (Levels 2 and 3), and for even weaker students, they refer to qualitative or non-numerical judgment (Level 1). This model is presented in Table 2.

Table 2

Synthesis of the developmental model of Clark and Kamii (1996) (Bisaillon, 2021)

Levels	Description of the thinking
I Qualitative judgment only	Consider only the length of the fish to decide the amount of food. This level of thinking could be described as non-numerical; rather, it is a visual comparison based on the size of the fish.
II Additive thinking using +1 or +2	Plus one counter, if the fish is a little larger or plus two counters, if it is much larger.
III Additive thinking taking into account +2 and +3	Using the information given for multiplication, but only doing an addition.
IV - A Beginning of a multiplicative thinking	Successful only after questioning by the interviewer.
IV - B Multiplicative thinking	Spontaneous success.

According to their results, multiplicative thinking begins to be present in a significant way in grade 2. However, only 49% of fifth graders have a "solid" multiplicative thinking. Their results are consistent with those of Jones and al. (1994, 1996).

On their side, Jacob and Willis (2003) conducted a review of the literature that allowed them to suggest four phases in the development of multiplicative thinking. They wanted to better understand how children transitioned from additive to multiplicative thinking. So, to study this evolution, they described different components associated with additive thinking and others associated with multiplicative thinking. Their work on multiplicative thinking suggests similar levels of development than Clark and Kamii (1996), moving from one-to-one counting, using additive components, through the beginning of group counting to an understanding of multiplicative relationships. These phases are briefly summarized in Table 3.

The levels proposed by Jacob and Willis (2003) are similar to those of Clark and Kamii (1996). It seems that before acquiring multiplicative thinking, children develop additive thinking. To better understand this additive thinking, research on counting was reviewed, particularly those associated with counting strategies based on global recognition. According to Brissiaud (2005), children must use two modes of counting: sequential counting, i.e., enumeration, or global visual perception. Furthermore, Twomey and Dolk (2011) state that in order to achieve grouping which is the focus of the present study, students must forget these one-to-one counting strategies. Therefore, it is the studies associated with global counting will be discussed in the following. Starkey and McCandliss (2014) found that items organized in small groups of 2 or 3 were counted faster than disorganized items. They named *groupitizing* this ability to count better when items are placed in small groups. This ability allows the use of global counting, i.e. a block processing of the figures, without going through enumeration, as when one counts the dots on a dice, just by glancing. A quantity is associated with patterns that will become increasingly familiar. This form of quantity processing leads to conceptual subitizing (Clements, 1999). Conceptual subitizing involves finding the total of a collection from the composition or decomposition of a constellation. The better the child's ability to compose and decompose a quantity, the more flexible he or she is in this activity and the more likely he or she will be able to acquire a good "mental representation" of quantities, a good "mental photograph" (Bergeron, 2003, p. 15). According to some authors, students who perform better in groupitizing and conceptual subitizing also perform better in numeration (Starkey & McCandliss, 2014; Schindler et al., 2020).

Table 3

Jacob and Willis (2003) phases of development of multiplicative thinking

Phases	Description of the thinking
1 One-to-one counting	Enumeration of each of the elements to solve the problem
2 Additive composition	Use of additive components to solve problems
3 Manny-to-one counters	Being able to count the number of groups and the number in each group
4 Multiplicative relations	Beginning to understand multiplicative relationships

Finally, groupitizing, on the other hand, is based on perceptual subitizing (Clements, 1999; Clements et al., 2019). Several studies have been conducted to better understand what is subitizing. For the present research, it is considered to be the instantaneous perception of small quantities (0-3) (Baroody, 2004; Dehaene, 2003; Houdé, 2004; Starkey & Cooper, 1980; Wynn, 1992a, 1992b). It is present in infants and is associated with the first manifestations of number sense.

Using Jones et al.'s (1994, 1996) developmental model of multi-number system understanding as a starting point, and adding other studies to complement it, we have presented studies that examine the development of number sense from infancy through numeration comprehension. The next section proposes a synthesis of these studies and leads to the proposal of a developmental hypothesis for number sense.

Synthesis of studies: Number Sense Development Hypothesis

It seems that number sense is associated with understanding number and number system, and with flexibility in manipulating numbers. It is a way of thinking (Reys, 1994). This understanding is associated with the ability to build an internal pictorial and dynamic representation of quantities. Based on the researches that have been presented, it has been possible to identify key elements leading to numeration understanding. However, no research linking all of these elements together has been identified. Some studies, such as those by Jones et al. (1994), Clark and Kamii (1996), or Jacob and Willis (2003) present developmental levels, but only for some part of number sense. Some others, on the other hand, proposed learning trajectories for certain aspects of number sense and not a learning trajectory for number sense (Clements & Sarama, 2004, 2009). The definition of a continuum would make it possible to integrate all these researches into a single coherent model.

Therefore, the synthesis of the research identified in the theoretical framework led to a hypothesis for the development of number sense and the mental representation of number. It is presented more like waves than linear development considering that all cognitive development occurs gradually and not necessarily in a linear way (Siegler, 2010). Moreover, since what we are interested in is the development of number sense, it implies that "number" will not necessarily have the same characteristics throughout this development. The purpose of this continuum is the understanding of number and numeration through the construction of mental representations, and it proposes the development of this understanding. Five levels of development were established. A possible age was associated with each level. A phase in the development of the child's number sense and arithmetic thinking has also been associated with each level. Procedures used by students that could be associated with these mental representations in mathematical tasks are suggested. Flexibility, an important component of number sense, is present at all levels. The hypothesis for the development of number sense and mental representations of number is presented in table 5.

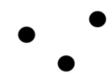
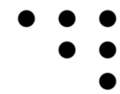
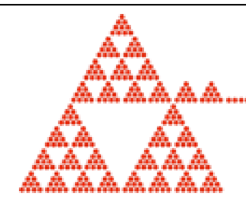
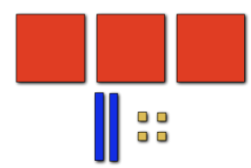
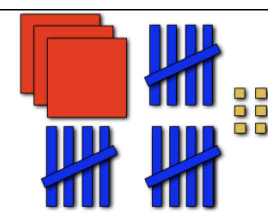
The first two levels could be associated with the pre-place value of Jones et al. (1994, 1996) and the qualitative judgment of Clark and Kamii (1996).

The first level provides information about the number sense of infants up to the age of 4. Young children can instantly recognize three or four elements or less (Baroody, 2004; Dehaene, 2003; Houdé, 2004; Starkey & Cooper, 1980; Wynn, 1992a, 1992b). To do so, they use their subitizing skills. The mental representation of quantities is not necessarily organized. It may be associated with a cluster of three points or less. Around the age of 4, these skills develop and include counting larger collections, which leads to the second level. The transition between level one and level two may correspond to the first phase identified by Jacob and Willis (2003).

The second level would refer to number sense developed by 5- and 6-year-olds. It would be associated with the development of additive thinking, as proposed by Clark and Kamii (1996) and Jacob and Willis (2003). It is characterized by the ability to use groupitizing and conceptual subitizing skills (Brissiaud, 2005; Clements, 1999; Clements & Sarama, 2004, 2009; Starkey & McCandliss, 2014). Students will develop additive flexibility by counting globally quantities organized in familiar or unfamiliar constellations. They can count quantities up to 20 and mentally represent them. This mental representation is made of small groups added together. The quantities to be processed will become more and more important and a new need to organize them will be necessary. The transition between level two and level three may correspond to the third phase identified by Jacob and Willis (2003).

Table 5

Hypothesis for the development of number sense and mental representations of number (Bisaillon, 2021)

Lev.	Age	Number sense dev.	Evolution of the mental representation	Examples of the mental representation
1	0-4	Perception of small quantities	Students rely on their perceptual subitizing skills to mentally represent small quantities (1 to 4 elements).	
2	5-6	Additive thinking	Students use their adding skills to mentally represent collections of up to 20 items; they use groupitizing and conceptual subitizing. They demonstrate additive flexibility.	
3	6-7	Multiplicative thinking Pre-position value	Students perceive the relevance of grouping. They can form mental representations of quantities organized in equal groups. They demonstrate multiplicative flexibility.	
4	7-8	Passage to tens Passage to hundreds	Students develop representations of groups of tens to facilitate their counting. They understand the role of hundreds; they can manipulate this group of groups. They understand the concept of equivalence. They demonstrate flexibility in equivalence.	
5	8	Understanding place value	Students manipulate numbers and their multiple representations. They demonstrate operational flexibility.	 **3c + 15d + 6u = 456**

Level three of the continuum appears to develop around 6 or 7 years of age. It would correspond to multiplicative thinking as defined by Clark and Kamii (1996). It is the beginning of understanding multiplicative relationships (Jacob & Willis, 2003) and the beginning of group counting (Jones et al., 1994). Students now understand the role of grouping in the number system (Bednarz & Dufour-Janvier, 1984a, 1984, 1986; Koudogbo et al., 2017). They will be able to mentally represent grouped quantities (organized into groups of groups). They will develop multiplicative flexibility. The mental representations would be organized in groups and groups of groups. Then, groups and groups of groups will get bigger and bigger. To facilitate the processing of quantities, the use of a system will become necessary. Understanding this system begins at the next level.

The fourth level appears to develop around ages 7-8. It would be the transition to tens in the first place, and then to hundreds (Jones et al., 1994). According to these authors, at this level, children can use materials representing groups of tens and hundreds. They can represent quantities concretely with base ten blocks, for example. Thus, the mental representation could be associated with base ten blocks, well organized. They can also compose and decompose a number, represented with base ten blocks, in different ways. They develop representational flexibility. The system is increasingly mastered by students. The links between the different representations remain to be solidified. The awareness of a better representation in comparison with another one to solve a problem more easily remains to be done. These are the main challenges of the next level.

Level five appears to develop around age 8. It would be the understanding of the place value of the number system (Jones et al., 1994). According to these authors, students at this level easily manipulate different quantities. They can do this by using concrete or symbolic representations of quantities. They manipulate number representations both with external representations and internal representations (Bednarz & Janvier-Dufour, 1984a, 1984b, 1986; Hiebert & Wearne, 1992; Thomas et al., 2002; Reys, 1994). They can compose and decompose numbers easily, according to the task they must accomplish (Reys, 1994). This is an important distinction from Level 4. The mental representation could be associated with base ten blocks. It could also be associated with any other material used to represent quantities concretely. Students can manipulate the blocks in their mind, composing and decomposing the numbers and choose the best representation to solve the problem.

This hypothesis of developing number sense emerged from the studies consulted. It served the purpose of this article, which was to better understand how number sense develops and to propose a continuum of development. This continuum could be used as a reference framework to create and organize teaching sequences according to these developmental levels. This is discussed in the conclusion.

Conclusion

To address the research objective of better understanding how number sense develops, an analysis and synthesis of existing literature was conducted. First, number sense was associated with understanding numbers and numeration, flexibility with number manipulation, and mathematical thinking. From this conception of number sense, a synthesis of the literature was carried out. Since no continuum of number sense development from infancy to the understanding of numeration was identified in this review of the literature, studies focusing on points in this development were selected. These studies focused on either the understanding of number and numeration, or the development of flexibility in number manipulation, or the development of arithmetic thinking. From the synthesis of these studies, a developmental hypothesis, associated with the construction of dynamic and pictorial mental representations, was proposed. This hypothesis suggests five phases, starting from what toddlers seem to understand about number to the understanding of numeration. They have been named: perception of small quantities, additive thinking, multiplicative thinking and pre-place value, passage to tens and hundreds and understanding place value. This developmental hypothesis may have implications for theoretical research in didactic of mathematics as well as for practical and professional work. These implications are discussed in the following paragraphs. Finally, future perspectives associated with this developmental hypothesis are presented.

One main contribution of this research to the advancement of scientific knowledge is the intersection of research from cognitive sciences and didactic. These two fields should enrich each other (Fayol, 2012). Indeed, studies in cognitive science were considered to better understand children's subitizing and groupitizing skills (Starkey & McCandliss, 2014; Wynn, 1992a, 1992b). Studies in didactic were referred to when it came to understanding multiplying thinking and numeration (Bednarz and Janvier-Dufour, 1984a, 1984b, 1986; Clark & Kamii, 1996; Dyson et al., 2015; Hiebert & Wearne, 1992; Jones et al., 1994, 1996; Jordan 2010; Jordan et al. 2016; Thomas et al., 2002). The number sense development hypothesis considers studies from these two streams of research. To our knowledge, such a model does not currently exist in scientific literature, either in cognitive science or in mathematics. However, such a model is necessary to guide future work on this issue.

In terms of practical and professional outcomes, the hypothesis of the development of number sense is a source of outcomes. According to some authors (Barabé, 2011; Mary & Squalli, 2021), it is important to build on the mathematical potential of students. One way to achieve this goal is to think of mathematics education in terms of cognitive pathways. Indeed, if teachers based their teaching on such a continuum, the activities offered to students would follow a cognitive pathway (Barabé, 2011; Mary & Squalli, 2021). The influence of such a continuum would be even stronger if teachers built on it as early as kindergarten. It would be possible to address each of the key concepts in the right order, which would have the potential to increase student success. Thus, syntheses presented in this research could help in the development of learning tasks and learning assessment tools that will support students developing their number sense, particularly for students experiencing difficulties. A better understanding of how a number sense develops could also provide input to the reflection about learning trajectories. Indeed, taking an interest in this development allows to have an overview that learning trajectories do not provide. This overview could support these trajectories, somewhat as the spinal column does in the human body; learning trajectories would then be the elements that are connected to this spinal column.

Finally, it would be interesting to continue the reflection on the development of number sense that has been established. One way is to use and to study this hypothesis to set up learning activities based on it. This was done in a thesis (Bisaillon, 2021). Considering the conditions to promote the development of number sense, a developmental research that led to elaborate an assessment tool and a teaching sequence was conducted. These tools were based on the hypothesis of the development of number sense. The in-context viability of these tools was assessed by thirteen education professionals. They commented, criticized, and offered suggestions to facilitate their use in real classroom contexts. Another perspective for the future would be to complete this hypothesis of the development of number sense by including the other types of mental representations that students can construct in relation to numbers, i.e. symbolic, verbal and emotional representations. It would also be interesting to try out this hypothesis with preschool and elementary school students, who may or may not have learning difficulties.

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