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A Unique Experience Learning Calculus: Integrating Variation Theory with Problem-Based Learning

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Abstract: The paper proposes a pedagogical approach to teaching and learning calculus differentiation formulas that synthesizes the principles of variation theory (VT) and *bianshi* in a problem-based learning (PBL) format. Unlike traditional approaches that view formulas procedurally, the paper adapts Steinbring's (1989) distinction between "concept" and "symbol," abstracting differentiation calculus formulas as "concept" (i.e., the meaning of the formula) and "symbol" (i.e., procedural knowledge about how to apply the formula). The paper then aligns this distinction with VT and *bianshi* pedagogies. While VT emphasizes more static elements of conceptual knowledge (e.g., highlighting the contrast between conceptual and non-conceptual features of the object of learning), *bianshi* broadens the concept of variation, offering more dynamic principles of variation through procedural variation (e.g., via the process of problem solving) (Gu et al., 2004). Combining VT and *bianshi* into a single pedagogical application yields an eight-step approach to teaching and learning calculus differentiation formulas.

Keywords: *bianshi*; calculus; problem-based learning; variation theory.

Introduction

Calculus is an important subject in students' academic careers because those who succeed in the subject "have the entire field of mathematics open up to them at the postsecondary level" (Bressoud, 2021, para. 3), making careers in engineering, the biological sciences, physics, and other fields possible (Bressoud, 2020). Yet it is common to find 25% to 30% of college students failing calculus, forcing them to seek a career that does not require calculus (Bressoud, 2021). In addition, access to calculus is arguably inequitable, with only 7% of high school students ranking in the bottom quartile of socioeconomic status taking calculus, in contrast with 38% of high school students ranking in the top quartile of socioeconomic status taking calculus (Bressoud, 2021).

These problems persist despite past attempts to reform calculus, such as the reform movement initiated by the National Council of Teachers of Mathematics (NCTM) in its 1989 publication of the Principles and Standards for Mathematics Education. Under this reformed calculus, less emphasis is placed on proofs and more on understanding what the proofs mean, and, relying on group work more than lecture, the reformed calculus encourages students to construct their own meaning of the concepts they learn and assesses them on projects in which they explain why a solution to a problem is correct rather than provide the solution (Murphy, 2006). By contrast, traditional calculus is highly rigorous, placing heavy emphasis on theorems and proofs, much memorization, and, under traditional calculus, the teacher is the main source of knowledge, delivering lectures while students take notes (Murphy, 2006).

Analyzing the views of 24 experts in the field, Sofronas et al. (2011) identified the following important themes for a first-year calculus course: mastery of fundamental concepts and/or skills (100%), relationships between and among concepts and skills (83%), ability to use ideas of calculus (63%), and the context and purpose of calculus (42%). Differentiation formulas are at the heart of calculus, and the experts concur that derivative topics must be included in a calculus course (100%), and that among the essential components of derivative topics are derivative computations (67%), graphical representations of derivatives (29%), and rate of change (50%) (Sofronas et al., 2011).

Despite the importance of calculus in the high school and undergraduate curriculum, many students find calculus difficult, and studies have uncovered a number of misconceptions that students have about calculus concepts and obstacles to understanding that they face (Sofronas et al., 2011). If combined with a set of central objectives that define an understanding of first-year calculus, the knowledge of these misconceptions and obstacles could yield curricular modifications that foster student understanding (Sofronas et al., 2011).

The purpose of this paper is to help students conquer the challenges they face in calculus courses with an efficient approach to teaching and learning calculus, with a specific focus on differentiation formulas. The approach uses scaffolding to help students develop from their current abilities to their potential abilities, and does so through procedural variation (Gu et al., 2004). The paper proposes a sequence of problems that blend problem-based learning (PBL) with the variation theory (VT) of learning to present a comprehensive approach to teaching and learning differentiation formulas. Since the content of the paper is not widely known, the paper is not an empirical study. Instead, it is an explorative study to help educators understand how variation theory can be used to advance knowledge of calculus. In line with this objective, the paper proposes a unique way of using variation theory to learn calculus.

Understanding the Mistakes that Students Make in Calculus

Research on students' mistakes in calculus is vast, as studies suggest that students across the globe (e.g., Quezada, 2020; Ruslimin et al., 2019; Wewe, 2020), in various disciplines (e.g., Mokhtar et al., 2010; Quezada, 2020; Siti Fatimah, 2019), hold persistent misunderstandings and errors (Smith et al., 1993). The types of mistakes that students make range from careless slips to misunderstandings of important concepts (Olivier, 1989; Othman et al., 2018; Siyepu, 2013). Othman et al. (2018) reported that differentiation is one of the most difficult topics for students to master, and students frequently make mistakes in basic differentiation problems, having difficulty, for example, with symbols and formulas. These mistakes can be of many sorts, including "the wrong calculations, lack of conditions, misunderstanding concepts, misunderstanding theorems or formulas, incorrect memory, applying basic techniques wrongly, presenting solutions wrongly, not paying attention to assumptions of a problem, misconception, and not exhausting all possibilities as possible" (Othman et al., 2018, p. 2). Siyepu (2013) observed students make conceptual and procedural mistakes in derivatives of trigonometric functions, and, following Olivier (1989), identified such mistakes as slips, errors, or misconceptions. A slip is a careless, nonsystematic, easily corrected mistake that results from processing; an error is a regularly applied mistake that results from planning, and is a symptom of a flawed underlying conceptual structure; a misconception is a flaw in the underlying conceptual structure that gives rise to errors (Siyepu, 2013). According to Smith et al. (1993), misconceptions can be widespread, strongly held, and resistant to change.

Niss (2003) distinguished eight competencies in mathematics: (1) mathematical thinking, (2) posing and solving mathematical problems, (3) modeling, (4) reasoning, (5) representing, (6) handling mathematical symbols and formulas, (7) communicating, and (8) using aids and tools. The mastery of concepts in calculus is thus important because it complements procedural mastery (the ability to handle mathematical symbols and formulas) and because it helps students understand and solve calculus problems, including real-life problems (Wewe, 2020). Yet research suggests that students have difficulties understanding concepts in basic calculus (Sofronas et al., 2011). Wewe (2020) analyzed performance on a second-semester calculus midterm exam and interviewed 11 students of the Mathematics Education Study Program at STKIP Citra Bakti, who had previously completed the first-semester calculus course. Results of the study revealed that students had difficulty understanding basic calculus concepts, including absolute value, the limit theory, and the properties of the algebraic derivatives function (Wewe, 2020). Ruslimin (2019) echoed Wewe's findings when he conducted a study that examined the number of errors that the lowest performing students made related to second-semester calculus material in the Mathematics Education Study Program at the STKIP Muhammadiyah Enrekang. Ruslimin (2019) found that these students did not understand the concept of derivatives, which in turn hindered their understanding of integration. According to Orton (1983) and Siyepu (2015), students have particular difficulty understanding the meaning of the derivative if it is written as a fraction.

What Are the Causes of Students' Mistakes in Calculus?

Research suggests multiple causes of students' poor performance in calculus. Several classifications have been proposed to characterize these difficulties. For example, Quezada (2020) and Socas (1997) divided the difficulties into five categories: (1) difficulties related to mathematical objects, (2) difficulties related to mathematical thinking process, (3) difficulties related to mathematics teaching processes, (4) difficulties related to students' cognitive processes, and (5) difficulties related to students' attitudes toward mathematics. Furner and Duffy (2002) found that many American students have negative attitudes toward mathematics, and Pyzdrowski et al. (2013) found that, after controlling for high school GPA and calculus readiness measurements, attitude was the strongest predictor of performance in a calculus course.

A study by Quezada (2020) of three competence-based engineering programs at a Chilean University suggested that insufficient basic knowledge of functions and algebra is a key source of poor student performance. Similarly, Luneta and Makonye (2010) asserted that "students' performance in calculus is undermined by weak, basic algebraic skills of factorization, handling operations in directed numbers, solving equations, and poor understanding of indices" (p. 167). Again, Othman et al. (2018) found that students made mistakes in basic differentiation formulas due to insufficient mathematical knowledge and being overly focused on specific mathematical rules.

Students' lack of motivation and insufficient basic mathematical skills are only part of the picture. Other reasons students make mistakes and find calculus difficult lie within the realm of teaching. Although the causes of teaching shortcomings in calculus are complex and merit more extensive analysis, most research on teacher shortcomings focuses on the knowledge of teachers and notes that some teachers' knowledge of concepts in calculus falls short (e.g., Eichler & Erens, 2014; Fothergill, 2011; Toh, 2009). Teacher shortcomings, however, are of various sorts and include "limited concepts, incorrect delivery of concepts, and minimum implementation of concepts" (Wewe, 2020, p. 7).

Why Problem-Based Learning (PBL) Is Useful in Calculus

The learning and teaching approach that this paper develops draws extensively on the problem-based learning (PBL) modality. Thus it is important to understand the merits of the inclusion of this approach in the current framework. The five chief characteristics of PBL are (1) the use of ill-structured problems to begin the process of learning, (2) the use of small groups for collaborative learning, (3) student-centered rather than teacher-centered learning, (4) the facilitating role of teachers, and (5) self-study (Barrows, 1996; Wijnia et al., 2019). Ill-structured problems include cases, stories, or phenomena in need of explanation (Barrows, 1996), and they lack clearly stated goals and can have more than one solution or more than one path to a solution (Jonassen, 1997). Given such a problem, PBL goes through three phases: "(a) an initial discussion phase in which the problem is defined and hypotheses are generated, (b) an information gathering and self-study phase, and (c) a debriefing or reporting phase" (Wijnia et al., 2019, p. 274).

According to a traditional approach, teachers are viewed as authorities and transfer knowledge typically in a lecture-based format, in which students passively assimilate information (Mokhtar et al., 2010). PBL, by contrast, assumes self-directed learning, in which the teacher's role is a facilitator (Moust et al., 2021). Research suggests that student achievement in calculus is higher in PBL than in a traditional setting, and that PBL improves critical thinking and problem solving skills (Elshafei, 1998; Gallagher, 1997; Mokhtar et al., 2010). Mokhtar et al. (2010) examined the effect of PBL on students' achievement and engagement in a basic calculus course at a private university. Derivatives were covered during the two weeks of the study. Two groups were compared: a control group (n=24) that used a traditional method of teaching calculus, and the experimental group (n=24) that used a PBL approach. Sources of data were rubric engagement, a questionnaire with open-ended questions, and a post-test (Mokhtar et al., 2010). The results suggest that students enjoyed group work and became more confident in their abilities, and high correlation scores on engagement and performance suggest that PBL is a valuable tool in teaching calculus. Thus PBL may address one of the causes of students' problems with calculus—lack of motivation and interest in the subject.

Similarly, Rézio et al. (2022) found that PBL methodology used for calculus learning in online classes enhanced engineering students' awareness of engaging in scientific discovery. Kattayat and Josey (2019) found that, in a calculus-based college physics course that used a technology-based online learning platform, a PBL format significantly increased students' conceptual understanding of work-energy and velocity problems. In another study of PBL in a calculus-based college physics course, Simangunsong and Parsaoran (2021) likewise concluded that a PBL format increased students' learning.

To address persistent misconceptions and errors that students possess when studying calculus, one must, however, look more deeply at the structure of knowledge that students acquire and determine whether some characteristics of such structures impede students from forming a correct understanding of concepts and processes in calculus. To explore these issues, one must look more deeply into the process of mathematical knowledge acquisition that students possess when solving calculus problems. Research suggests that students experience difficulty separating definitions and concepts (Tall & Vinner, 1981). The problem of passive learning is exacerbated in subjects that students find more difficult, such as calculus, in which students develop incomplete or misleading concept images of derivatives that conflict with the formal rules of derivatives and their differentiations, and in which students have limited opportunities to revise their corrupt concept images (Tall & Vinner, 1981). Active learning combined with social experiences (e.g., group work), critical thinking, and problem solving are incorporated into PBL and are cardinal features of a reformed-based curriculum that is buttressed by constructivist learning theories (Mokhtar et al., 2010).

Research suggests that, in general, students find calculus a difficult subject (Bressoud, 2021; Sofronas et al., 2011), and they commit errors and hold misconceptions in both the procedural and conceptual domains of knowledge (Othman et al., 2018). Yet research also suggests that conceptual knowledge may facilitate acquisition of procedural skills (Chappell & Killpatrick, 2003). Thus the framework that is presented in this paper focuses on building connections between conceptual and procedural knowledge, as both of these types of knowledge are essential and inseparable when discussed in the context of differentiation formulas.

Although PBL has substantial benefits in mathematics learning and teaching, critics of PBL point out potential drawbacks of the approach: (1) poorly designed problems may negatively affect students' motivation to learn (Hung, 2019; Hung et al., 2013); (2) insufficient content may be covered, because some topics are covered more quickly (Albanese & Mitchell, 1993; Hung, 2019) and because student self-directed learning may not result in sufficient content knowledge compared to faculty intended learning objectives (Hung, 2019; O'Neill, 2000); (3) although some studies report that students' motivation increases when using PBL (e.g., Moust et al., 2005; Romito & Eckert, 2011), generalizing these results to long-term effects may be problematic, as a novelty effect may be present in the implementation of PBL in these studies (Hung, 2019).

The problems presented in the current study have some characteristics of PBL, but are not representative of the PBL approach in its full sense. The notion of "problem" in PBL is one that may not have a solution, or a single solution, and serves more as a facilitator of the process of inquiry and discovery (Wijnia, 2019). Oftentimes, PBL problems focus on explanation and the process of solving rather than on the solution. The ill-structured real-life problems that PBL prefers to use make it difficult to satisfy hierarchic, well-organized learning objectives, and this may help explain PBL's insufficient content coverage (Hung, 2019; Hung et al., 2008). One solution would be to integrate variation theory in the context of PBL to help students integrate content knowledge.

The Variation Theory of Learning—An Integrated Approach

Formulas and theorems are central to calculus, and students find them challenging to learn (Qi et al., 2017). Research suggests that implementing variation theory can help students grasp the essence of formulas and theorems (Yuan, 2006) as well as their conditions and conclusions (Wu & Liu, 2006).

Variation theory is a theory of learning that has been applied to the learning of mathematics (e.g., Kullberg et al., 2016; Kullberg et al., 2017). According to the theory, students learn mathematics through variation against a backdrop of invariance, and do so in four ways, which variation theorists call contrast, generalization, separation, and fusion (Marton, Runesson, & Pang, 2004; Marton, Tsui et al., 2004; Pang et al., 2017).

Contrast

One way students learn about a concept in mathematics is by contrasting it with—that is, by seeing how it differs, or varies, from—other mathematical concepts (Marton, Runesson, & Pang, 2004; Pang et al., 2017). According to variation theory, contrast is necessary for learning a mathematical concept, but it is not sufficient (Marton, Runesson, & Pang, 2004). In addition, students must learn through generalization.

Generalization

Students also learn about a mathematical concept by understanding how the various instances of that concept are similar to each other (Marton, Runesson, & Pang, 2004). Variation theorists call this approach to learning generalization (Marton, Runesson, & Pang, 2004; Pang et al., 2017).

Separation

To learn about a mathematical concept through separation involves varying a critical aspect of that concept while keeping other aspects constant, or invariant (Marton, Runesson, & Pang, 2004; Pang et al., 2017).

Fusion

To learn about a mathematical concept through fusion involves varying more than one critical aspect of the concept at the same time, thus helping the student understand how the varied aspects are related (Marton, Runesson, & Pang, 2004; Pang et al., 2017). Calculus-related examples of contrast, generalization, separation, and fusion can be found in Table 1 at the end of the explanation of the (VT + PBL) model.

Qi et al. (2017) applied variation theory in the context of learning algebraic formulas (e.g., quadratic formulas), specifically adopting ideas taken from the *bianshi* framework of Gu et al. (2004). *Bianshi* is similar to, yet distinct from, variation theory. Like variation theory, *bianshi* proposes that learning takes place through variation against a background of invariance. Unlike VT, *bianshi* distinguishes conceptual from procedural variation. Conceptual variation is the use of multiple perspectives to understand concepts, e.g., the use of different visual examples or a comparison with nonstandard examples (Qi et al., 2017). Procedural variation, by contrast, is the teaching of process-oriented knowledge, i.e., the knowledge of how to do something, and it includes the scaffolding of problem solving from easier to more difficult problems (Qi et al., 2017). While conceptual variation is similar to the types of variation found in variation theory (e.g., generalization and contrast patterns of variation), which are more static, procedural variation is more dynamic (Gu et al., 2004). Developing this distinction, Qi et al. (2017) posit four types of variation that can be used in the teaching of algebraic formulas, with "variation in the style of examples or questions" (p. 130) and "variation in the way of recognizing the formula" (p. 130): (1) "seeing the similarity in concrete examples" (p. 130), (2) "acquiring a deeper understanding in application" (p. 130), (3) "analyzing the relationship between formulas" (p. 130), and (4) "recognizing the formula in multiple ways" (p. 130). Based on these distinctions, Qi et al. (2017) derived learning steps to be applied in planning a lesson.

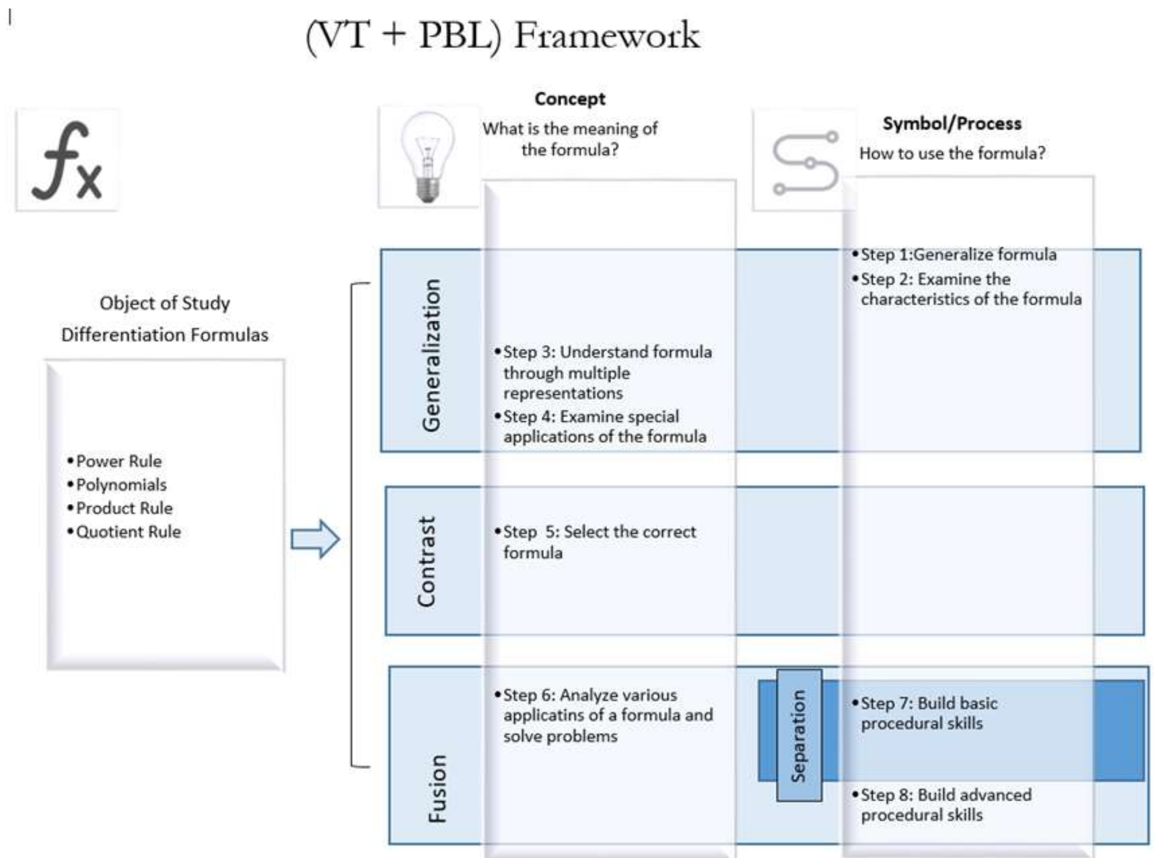
An Integrated Model of Teaching and Learning Differentiation Formulas

Variation theory—with its patterns of generalization, contrast, separation, and fusion—offers some guidance for how to teach derivative formulas, but the theory does not provide a formal structure for the sequence and format of learning activities. A variation theory approach to teaching differs from a conventional approach in that a conventional approach focuses on an "explain-practice-memorize" technique with random variation within any specific patterns in the examples (Ting et al., 2018). Thus using variation theory when teaching differentiation formulas is in line with the reformed calculus approach, as it focuses on a learner-centered approach to teaching by carefully constructing different types of variation that enhance conceptual understanding of mathematical knowledge. Likewise, although Qi et al.'s (2017) model of teaching formulas that is based on the *bianshi* theory of variation is useful in the context of algebra, transferring this approach to teaching calculus (e.g., differentiation formulas) may be problematic. First, the concept of a derivative is more complex than algebraic concepts, as it involves analysis not only of the original function, but also its derivative. Second, the procedures for taking a derivative, especially when complex functions are involved, is not intuitive, and may involve a confluence of algebraic steps. If one adopts a PBL modality for teaching differentiation formulas, even more care should be taken when selecting problems for complex topics, such as differentiation formulas. Thus, to embrace fully the potential of both the *bianshi* theory of variation and VT in the context of differentiation formulas, the author of this article proposes a framework that incorporates components of these theories in a PBL format through the linkage of object-symbol-concept relationships proposed by Steinbring (1989). Steinbring (1989) criticized the approach of teaching concepts when the symbol/sign is collapsed with the object by focusing too much on memorization and routinization of processes. By focusing on an epistemological triangle, in which procedural and conceptual knowledge are linked to the means of representation and of activity, building new knowledge by focusing on both meanings and procedures is necessary for a meaningful conceptual understanding in mathematics (Steinbring, 1989).

This paper adopts Steinbring's (1989) distinction between conceptual and procedural knowledge, applying it to knowledge acquisition of differentiation formulas via concept and symbol bifurcation. The integrated (VT + PBL) Framework for teaching and learning differentiation formulas is depicted in Figure 1.

Figure 1

Overview of (VT + PBL) Framework



Note: The process icon in Figure 1 is taken from Jimo Icons (n.d.).

While the two domains may intersect (Steinbring, 1989), in the context of the present framework with the focus on formulas, "concept" refers to the meaning of the formula (i.e., a conceptual understanding of the formula), and "symbol" refers to procedural knowledge about how to apply the formula. Thus "concept" and "symbol" are viewed as separate constructs when applied to differentiation formulas. The *bianshi* and VT theories help guide the process of learning through PBL, systematically following an eight-step approach. The two variation theories, *bianshi* and VT, are complementary in the context of differentiation formulas, because VT focuses on more static elements of conceptual knowledge (e.g., highlighting the contrast between conceptual and nonconceptual features of the object of learning (Gu et al., 2004), while *bianshi* broadens the concept of variation, offering more dynamic principles of variation through procedural variation (e.g., via the process of problem solving) (Gu et al., 2004). Activities within the steps and between the steps follow *bianshi* principles of procedural variation, which enhances understanding of the concepts of differentiation formulas (Gu et al., 2004). The following elements of procedural variation are present in the current model: (1) understanding differentiation formulas through a progression from concrete to general, (2) varying symbolic presentations of variables and parameters, (3) varying a problem, (4) using multiple methods to solve a problem, and (5) using multiple applications of a method (Gu et al., 2004).

Explanation of the (VT + PBL) Model

To generalize a formula from both conceptual and procedural perspectives, the learner examines various examples of the formula in different types of representations. In the PBL framework, the learner is guided to create a formula through a series of problems, affording an opportunity for the learner to examine applications of the formula in multiple contexts. Steps 1 through 4 allow the learner to experience a generalization pattern of variation to establish procedural and conceptual knowledge of the differentiation formula.

Step 1: Generalize the Formula

To understand the process that the formula conveys, in this step the learner progresses from specific to general examples to discern the formula. Using a PBL approach through a series of questions about specific examples, the learner establishes a general formula.

Step 2: Examine the Characteristics of the Formula

After establishing the formula, the learner examines the components and characteristics of the formula from a symbolic perspective to understand how the components of the formula impact the process that the formula represents.

Step 3: Understand the Formula through Multiple Representations (Graphing)

To gain a deeper understanding of the conceptual underpinnings of the formula, the learner is exposed to various contexts in which the formula is applied (for example, through graphical representations).

Step 4: Apply the Formula to Various Types of Functions

To strengthen conceptual understanding of the formula even further, the learner is asked to apply the formula in different contexts that may have some extensions of the rules, or that may feature some caveats that the learner may find challenging to overcome. Research suggests that variation theory helps students in concept construction (Marton & Pang, 2013; Ting et al., 2018). Thus experiencing various contexts in which differentiation formulas can be applied will help the learner understand the concept underlying the formulas more deeply.

Rationale for the Sequence of Steps 1 through 4

While Steps 1 through 4 provide the learner with the experience of establishing both conceptual and procedural understanding of differentiation formulas through a prescribed sequence of VT patterns of variation (Marton, Tsui et al., 2004), the sequence of steps from procedural to conceptual is also grounded in research on how students learn calculus. For example, Gray et al. (2009) point out that, when studying the concepts of limit, derivative, and integral, calculus students need to have a grasp of algebraic variables as generalized numbers and varying quantities, and they suggest that teachers of calculus should emphasize “the differing uses of variables in various contexts and strive to develop students’ conceptions of variables as changing and co-varying quantities” (p. 71). Furthermore, as Maciejewski and Star (2016) and Rittle-Johnson et al. (2015) point out, conceptual and procedural knowledge in calculus are intertwined, with increases in one leading to increases in the other. However, while conceptual knowledge leads to procedural knowledge and vice versa (Schneider et al., 2011), the teaching of procedural knowledge should emphasize the concepts underlying the procedures so that conceptual knowledge can develop simultaneously (Canobi, 2009; Maciejewski & Star, 2016; Siti Fatimah, 2019).

In addition, Step 3 focuses on multiple representations of differentiation formulas with specific emphasis on graphical representation. This approach is not only in line with a reformed approach to teaching calculus (as opposed to traditional), but is also rooted in research on how students learn calculus. Weber and Alcock (2004) found that students often confine their mathematical reasoning to one representational system, focusing, for example, on procedures rather than concepts. In support of this contention, Jukić and Dahl (2010) conducted a study showing that students remembered the procedure for finding the local extreme values of a function, but, unable to connect derivatives with the shape of the function, did not understand the concepts underlying the procedure. Studies have found that the concept images students have of functions and derivatives emphasize the use of formulas and rules, suggesting that their understanding of functions and derivatives is enhanced when multiple representations are present (Ferrini-Mundy & Graham, 1994; Habre & Abboud, 2006; Maharaj, 2013; Orton 1983).

In Steps 1 through 4 the learner is induced to apply algorithms and concepts of the formulas systematically to arrive at a correct solution. After the learner establishes a general understanding of the formula, that knowledge may not be sufficient to establish various facets of applications of the formula. Additional context is required to help the learner discern critical aspects of the formula and their application in more nuanced ways. Thus, the next step proceeds with contrasts to help the learner understand correct and incorrect ways of using the formula, still focusing on conceptual aspects of differentiation formulas.

Step 5: Select the Correct Formula and Apply It

In this step, the learner is asked to select the correct formula among a list of formulas, and the learner is exposed to the contrast pattern of variation. The context in which the formula is placed is more nuanced and complex and helps further refine the learner’s understanding of the formula and how to apply it.

Although generalization and contrast are essential to understand how to use differentiation formulas properly, so far the experience with formulas is limited. Separation and fusion will help the learner acquire a deeper understanding of the application of formulas.

Step 6: Apply the Concept of the Formula in Various Contexts

Step 6 focuses on exposing the learner to various contexts in which the formula is applied. This step focuses on fusing the aspects of formulas that have been learned so far on a conceptual level. For example, the learner is exposed to different types of problems that require the simultaneous application of various skills learned about the formula. This step may also contain ill-structured problems, or problems that require learners to reexamine their mental images of the formula to determine whether incongruences with definitions exist (Steinbring, 1989). Moreover, various problems from STEM fields that have applications of differentiation formulas are presented in this step to help students connect calculus formulas with real life.

After the fusion pattern of variation has been experienced on a conceptual level, the learner may now focus on the separation and fusion patterns of variation on a symbolic (or procedural) level.

Step 7: Build Procedural Skills

This step allows the learner to focus on changing the value of some critical features of differentiation formulas. In the earlier steps, students had a chance to examine the characteristics of the formulas. However, to acquire procedural fluency in the application of a formula, students need exposure to additional exercises that allow the development of these skills. Without such exposure, students are likely to make mistakes such as those that Othman et al. (2018) observed, e.g., forgetting to reduce the exponent by one and incorrectly using the Power Rule even after they simplify expressions. In addition, specifically targeted patterns of variation under separation help the learner ease the cognitive load associated with learning new skills (Marton, Tsui et al., 2004).

Step 8: Build More Advanced Procedural Skills (Fusion)

After having had an opportunity to practice the separation pattern of variation, when some of the components of the formulas were fixed, the learner can in this step integrate all previous procedural knowledge in various combinations at a symbolic/procedural level.

Rationale for the Sequence of Steps 6 through 8

Following Steps 6 through 8 allows students to reexamine their beliefs about how differentiation formulas are applied. The distinction between Step 6 (the conceptual level) and Steps 7 and 8 (the symbolic/procedural level) helps avoid excessive algorithmization and develops the meaning of a mathematical concept based on explicit rules and procedures (Steinbring, 1989). Steps 6 through 8 also serve the important purpose of helping the learner experience different representations of concepts and processes to develop a full understanding of differentiation formulas. This approach is also rooted in research. According to Watson (2017), task design through carefully selected patterns of variation can implicitly guide learners to understand the properties of the object of learning. From a pedagogical perspective, using variation effectively can help students avoid misconceptions and errors when learning the rules of differentiation (Watson, 2017). Moreover, from the mathematical examples or problems, it should be evident whether the learner is expected to learn about different methods of solving problems or whether the learner should focus on different representations of concepts and processes, as mathematics cannot be construed as a mere collection of symbols or an application of inductive reasoning from examples (Watson, 2017).

For a summary of the eight steps, as well as examples of each step, see Table 1.

Table 1

Steps and Examples Using the Power Rule

Step Number and Name	**Brief Description**	**Examples**
Step 1: Generalize the formula for the Power Rule.	The learner progresses from specific to general examples to discern the formula.	1. For each of the following equations, find f'(x) using Formula 1: $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ a. $f(x)=x^2$ b. $f(x)=x^3$ c. $f(x)=x^4$ 2. Compare your answers for equations a through c in Question 1. What do you observe? Write down the patterns that you observe. 3. Create a general rule for f'(x) that you can deduce from the above examples. 4. Write down in mathematical notation your general rule that would hold for any n in $f(x)=x^n$. This rule is called the Power Rule for derivatives.
Step 2: Examine the characteristics of the formula.	The learner examines the components and characteristics of the formula from a symbolic perspective.	5. What are the components of the f'(x) formula? Which values change and which do not change?
Step 3: Understand the formula through multiple representations.	The learner is exposed to various contexts in which the formula is applied.	6. Sketch a graph for $f(x)=x^2$. Then on the same graph sketch a graph for f'(x). Compare and contrast these two graphs. How do the two graphs relate to each other? 7. Repeat the same steps for $f(x)=x^3$. Compare and contrast these two graphs. How do the two graphs relate to each other? 8. Use a graphing calculator or a computer program to create a graph for $f(x)=x^4$ and its derivative, and then create a graph for $f(x)=x^5$. How do the functions and their derivatives relate to each other? 9. Compare all four graphs to each other. What general rules do you observe?
Step 4: Examine special applications of the Power Rule formula. • f(x)=x and f(x)=c • f(x)=bg(x)	The learner is asked to apply the formula in different contexts.	A. Examine the functions: f(x)=x and f(x)=c 10. For each of the following equations, find f'(x) using Formula 1. a. f(x)=x b. f(x)=5 11. Verify your answers in the previous question using the Power Rule formula. 12. Sketch a graph for f(x)=x. Then on the same graph sketch a graph for f'(x). Sketch on a separate graph f(x)=5 and its derivative. Compare and contrast the graphs of the

		functions and their derivatives. How does the graph of the function and that of its derivative relate to each other? 13. Create a general rule for derivatives of the functions $f(x)=x$ and $f(x)=c$, where c is any number. B. Examine the function: $f(x)=bg(x)$ 14. Suppose f is a differentiable function and b is a constant, such that $f(x)=bx^n$. Find the derivative of $f'(x)$ using the rule you found in question 4. 15. Explain the characteristics of this new formula. What components change, and what components do not change? What is the rule according to which the changing components change their values? 16. Examine how the formula will look when $b>0$ or $b<0$, and when $n>0$ or $n<0$. 17. Verify your formula in question 14: find $f'(x)$ using Formula 1. Show all your steps. 18. Compare and contrast the two formulas for derivatives in question 4 and in question 14. How do these formulas relate to each other? 19. Use a graphing calculator or draw two graphs by hand for each of the following functions: $f(x)=x^2$, $f(x)=x^3$, and $f(x)=x^4$. Draw a graph of each derivative on each corresponding graph. Compare these two graphs. What patterns do you observe?
Step 5: Select the correct formula.	The learner is exposed to the contrast pattern of variation through selection of the correct formula among a list of formulas.	20. Can you apply the formula you found in question 14 for the derivative, $f'(x)$, for the following functions: a. $f(x) = 5^{2x}$ b. $g(x) = (5+2x)^2$ c. $h(x) = \sqrt{5+2x}$ d. $k(x) = 5+\sqrt{2x}$ 21. Can you apply the formula you found in question 14 for the derivative, $f'(x)$, for the following functions? Find derivatives for the functions for which you are able to find derivatives. a. $f(x) = 5^{2x}$ b. $m(x) = 2x^5$ c. $t(x) = 3x^{\sqrt{3}}$ d. $g(x) = (5+2x)^2$ e. $z(x) = \left(5+\frac{2}{\sqrt{x}}\right)^2$ f. $h(x) = \sqrt{5+2x}$ g. $w(x) = \frac{5x-2x^2}{x}$ h. $v(x) = \frac{(5x-2x^2)^2}{x^3}$ i. $k(x) = 5+\sqrt{2x}$ j. $l(x) = 5+\sqrt{\frac{2}{x}}$ k. $n(x) = \frac{1}{\sqrt{5+2x}}$
Step 6: Analyze various applications of the formula and solve problems.	The learner applies the formula in various contexts, fusing aspects of the formula that have been learned so far.	22. Suppose the derivative of a function is $f'(x)=3x^2$. Give examples of functions that could have this derivative. What conclusions can you draw? 23. Suppose that for $f(x)=bx^n$, where b and n are constants, $f'(10)=100$ and $f'(5)=25$. Find the values of b and n. 24. Suppose $f(5)=12$ and $f'(5)=24$. Find $g'(5)$.

		a. $g(x) = 2f(5)$ b. $g(x) = \frac{4}{f(5)}$ c. $g(x) = \frac{3}{\sqrt{f(5)}}$ d. $g(x) = -4\sqrt{f(5)}$ 25. The number of wild blueberries B in a field with area A, in square meters, is modeled by the following equation: $B(A) = 0.52A^{0.639}$. Find B'(50). Interpret the answer you found. (Adapted from Stewart et al., 2021, p. 147)
Step 7: Build basic procedural skills.	The learner acquires procedural fluency by changing the value of some critical features of the formula.	26. Find the derivatives of the following functions: $f(x) = x^2$; $f(x) = x^3$; $f(x) = x^5$ $f(x) = 5x^2$; $f(x) = 20x^3$; $f(x) = 67x^5$ $f(x) = -5x^2$; $f(x) = -20x^3$; $f(x) = -67x^5$ $f(x) = 1.5x^2$; $f(x) = 0.75x^3$; $f(x) = 200.75x^5$ $f(x) = -1.5x^2$; $f(x) = -10.75x^3$; $f(x) = -200.75x^5$ $f(x) = \frac{1}{2}x^2$; $f(x) = \frac{15}{21}x^3$; $f(x) = \frac{11}{230}x^5$ $f(x) = -\frac{1}{2}x^2$; $f(x) = -\frac{15}{21}x^3$; $f(x) = -\frac{11}{300}x^5$
Step 8: Build advanced procedural skills.	The learner fuses knowledge from the previous steps and is exposed to different types of problems that require the simultaneous application of various skills learned about the formula.	27. Find the derivatives. a. $f(x) = x^4$ b. $f(x) = 10x^{3.5}$ c. $f(x) = 1.5x^{25}$ d. $f(x) = 21\, x^{\frac{1}{7}}$ e. $f(x) = \frac{3}{5}x^{\frac{1}{3}}$ 28. For the above functions, find the derivatives at x=1, x=-2, and x=4.

Conclusion

Both VT and PBL offer some hope to students when applied independently of each other in the teaching of calculus. However, when combined, these two learning approaches present an exciting synergy that can be explored in future research. For example, future research can explore this synergistic potential in the context of further calculus topics, not just teaching and learning differentiation formulas, as is demonstrated in this paper. Moreover, it would be interesting to explore how to combine these teaching approaches when there is more emphasis on the acquisition of conceptual rather than symbolic/procedural knowledge (as the latter type of knowledge is more predominant in formula applications and teaching). Combining PBL and VT offers teachers an empowering tool that can help create customized experiences not only for different classes that exhibit different levels of mastery, but also for individual students by assigning different problems to different students (or groups of students). Thus in future research, it would be useful to explore the connections between VT and PBL from an empirical perspective to determine whether students' performance in calculus courses can be improved using that integrated framework.

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