



A Comparative Analysis of Classical and Contemporary Approaches to the Calculus of Tangents: A Conceptual Perspective

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Abstract: This study examines how the concept of a tangent line has been understood in classical and contemporary mathematics. Classical approaches interpret tangents through approximation, algebraic reasoning, geometry, and infinitesimals, while contemporary calculus defines them using derivatives and limits. This study uses a qualitative conceptual and historical analysis to examine how tangents are understood across these approaches. The analysis focuses on key ideas such as slope, change, approximation, and formal definition, and compares how these ideas are represented in classical and contemporary mathematics. The findings show that classical approaches emphasize intuitive and conceptual reasoning, while contemporary approaches emphasize formal definitions and symbolic procedures. The findings also indicate that these approaches provide complementary forms of mathematical thinking. The study concludes that understanding tangents requires connections between intuitive meaning and formal reasoning. The study contributes to mathematics education by highlighting the importance of integrating classical and contemporary perspectives to support deeper conceptual understanding of tangents.

Keywords: Calculus; Conceptual understanding; Derivatives; Historical approaches; Mathematical proficiency; Tangent line.

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Introduction

The concept of a tangent line is central in calculus and plays an important role in understanding change, motion, and relationships between variables (Bressoud, 2022; Edwards, 1979; Tall, 2012). In modern mathematics, a tangent is defined as the slope of a curve at a point, obtained using the derivative and limits (Bressoud, 2022; Rivera-Figueroa & Cruz-Canales, 2025; Spivak, 2018; Zill & Wright, 2009). This definition is widely used in teaching and applications. However, this formal view is the result of a long historical development, where earlier mathematicians approached tangents using different methods and ideas that did not rely on limits (Boyer, 2012; Cajori, 1897; Suzuki, 2005).

Historical studies show that the concept of tangents developed gradually through different approaches to understanding change (Boyer, 2012; Cajori, 1897; Grabiner, 1983). Several classical mathematicians made important contributions to the development of tangent methods (Bos, 1997; Edwards, 1979; Suzuki, 2005). Pierre de Fermat introduced the method of adequality, which used a form of comparison and approximation to determine tangents to curves (Jensen, 1969; Strømholm, 1968). René François de Sluse developed algebraic procedures that allowed tangents to be computed more directly for certain curves (Coolidge, 1951). Isaac Barrow approached tangents from a geometric perspective, emphasizing the relationship between curves and their slopes and providing ideas that later

influenced the development of calculus (Strong, 1970). Gottfried Wilhelm Leibniz introduced infinitesimals and the notation $\frac{dy}{dx}$, offering a new symbolic and conceptual way of expressing the slope of a tangent line (Bos, 1974; Roero, 2005). These approaches show that classical methods were diverse and grounded in different ways of thinking, including geometry, algebra, and intuitive reasoning.

In contrast, contemporary calculus defines the tangent line using limits and derivatives, emphasizing formal definitions and symbolic procedures (Bressoud, 2022; Rivera-Figueroa & Cruz-Canales, 2025; Spivak, 2018; Zill & Wright, 2009). This approach provides precision and generality, but it often focuses on rules and techniques (Tall, 1985; Tall, 2012). Research shows that students frequently understand tangents procedurally but have trouble explaining their meaning conceptually (Hogue & Scarcelli, 2022; Vincent et al., 2015).

These differences between classical and contemporary approaches highlight the need to revisit earlier methods of tangents (Katz, 1986; Suzuki, 2005; Zuccheri et al., 2013). Classical approaches offer alternative ways of thinking about slope and change, which can complement modern definitions (Grabiner, 2010; Kleiner, 2001; Tall, 2012). These methods provide valuable conceptual insights that remain relevant for understanding tangents. This paper is motivated by the need to better understand how the concept of a tangent line has developed over time and how different approaches shape mathematical understanding. By examining both classical and contemporary methods, the study aims to provide a clearer view of the conceptual foundations of tangents and to show how historical perspectives can enrich modern interpretations of calculus.

Problem Statement

In modern calculus, the tangent line is mainly taught using the derivative and limits (Grossman, 2014; Spivak, 2018; Zill & Wright, 2009). This approach is clear and powerful, but it often focuses on procedures and rules (Tall, 1985; Tall, 2012). Research by Grabiner (1983) shows that the concept of the derivative developed from earlier ideas about change and approximation, yet these earlier ideas are often not emphasized in teaching. Studies show that students can apply differentiation rules but struggle to explain the meaning of a tangent line conceptually (Hogue & Scarcelli, 2022; Vincent et al., 2015). Learners may develop a stronger conceptual understanding when they connect symbolic procedures with graphical representations, changing secant slopes, and the meaning of instantaneous change (Gilboa et al., 2019; Ozaltun-Celik, 2021; Vincent et al., 2015). Historical approaches to tangents are well documented and show the use of geometric, algebraic, and intuitive reasoning to understand change (Baron, 1969; Boyer, 2012; Cajori, 1897; Suzuki, 2005; Kleiner, 2001).

The gap in the literature is that there is limited conceptual comparison between these classical approaches and the contemporary derivative-based approach (Grabiner, 2010; Katz, 1991; Zuccheri et al., 2013). Most studies either focus on historical development or on modern teaching, but do not clearly analyze how the underlying ideas of tangents differ across these traditions, a concern also reflected in Gilboa et al. (2019). There is also limited attention to how the different representations found in classical and contemporary approaches may provide distinct ways of interpreting

slope, change, and tangency. If this gap is not addressed, the teaching and understanding of tangents may remain narrow and procedural (Hogue & Scarcelli, 2022; Tall, 2012; Vincent et al., 2015). Students may continue to apply rules without developing a deep understanding of slope, change, and tangency. This can limit their ability to connect different mathematical ideas and to appreciate the meaning of calculus concepts. To address this gap, this paper is guided by the following research questions:

- 1) How were tangents conceptualized in classical mathematical approaches?
- 2) How do contemporary derivative-based approaches differ conceptually from classical methods?

Literature Review

This section examines how the concept of a tangent line has been understood and developed in mathematics. It first focuses on classical approaches, exploring how mathematicians like Pierre de Fermat, René François de Sluse, Isaac Barrow, and Gottfried Wilhelm Leibniz conceptualized tangents using geometric reasoning, algebraic methods, and intuitive ideas of change. It then considers the contemporary understanding of tangents, where the concept is defined using derivatives and limits, with a focus on how this approach is interpreted in modern mathematics and teaching. By examining both perspectives, this section highlights how different approaches shape mathematical thinking.

Classical Approaches to Tangents

Classical approaches to tangents were developed before the formal definition of the derivative (Baron, 1969; Edwards, 1979; Edwards, 1994). These methods were based on geometric reasoning, algebraic manipulation, and intuitive ideas of change. Research by Boyer (2012) explains that early mathematicians focused on solving specific problems involving curves rather than building a general theory. Similarly, Cajori (1897) notes that these methods relied on practical techniques and reasoning rather than formal limits. This is illustrated in the following example, which demonstrates how tangents were determined using algebraic reasoning and approximation.

One of the earliest systematic approaches to tangents was developed by Pierre de Fermat through his method of adequality. This method involved comparing two expressions that were nearly equal in order to determine the slope of a curve (Edwards, 1979; Mahoney, 2018; Strømholm, 1968). Strømholm (1968) explains that Fermat introduced a small variation in the variable and then simplified the resulting expression to identify the tangent. Jensen (1969) shows that this method was effective for a wide range of curves and can be seen as an early form of differentiation. The following example demonstrates Fermat's method of adequality.

Example 1: Finding the Slope Using Fermat's Method

To obtain the slope of the tangent to $y = x^2$ at $x = 2$, using Fermat's Method, the following procedure is employed:

Step 1: Introduce a small increment: $x \rightarrow x + e$

Thus: $y = (x + e)^2$

Step 2: Expand

$$(x + e)^2 = x^2 + 2xe + e^2$$

Step 3: Compare with original

Original: $y = x^2$ and *New:* $y = x^2 + 2xe + e^2$

Step 4: Subtract the original expression from the new expression

$$(x^2 + 2xe + e^2) - x^2 = 2xe + e^2$$

Step 5: Divide the result by e

$$\frac{2xe + e^2}{e} = 2x + e$$

Step 6: Substitute 2 for x

$$m = 2(2) + e = 4 + e$$

As e becomes very small (approaches 0), the slope approaches 4.

Final Answer : Slope of tangent = 4

This example shows how Fermat's method determines the slope of a tangent using approximation rather than limits. René François de Sluse later introduced algebraic rules for finding tangents, especially for algebraic curves (Coolidge, 1951; Edwards, 1979; Kusraev & Kutateladze, 2017). According to Coolidge (1951), de Sluse's method provided a more direct procedure compared to earlier approaches, allowing mathematicians to compute tangents without relying heavily on geometric constructions. This marked a shift toward symbolic and algebraic reasoning in the study of tangents. To illustrate de Sluse's algebraic approach to tangents, the following example is presented.

Example 2: Determining the Tangent Using de Sluse's Method

For the curve $x^2 - 4y = 0$, the slope of the tangent at the point (2,1) is determined using de Sluse's approach as follows.

Step 1: *Identify the terms of the curve*

The terms are x^2 and $-4y$

Step 2: *Apply de Sluse's rule*

Each term is multiplied by the exponent of:

- x to form the numerator : $2x$,
- y to form the denominator: 4

Thus, the slope of the tangent is:

$$\frac{dy}{dx} = \frac{2x}{4} = \frac{x}{2}$$

Step 3: *Substitute $x = 2$,*

$$\frac{dy}{dx} = \frac{2}{2} = 1$$

Final Answer:

$$m = 1$$

This result highlights the use of direct algebraic rules in de Sluse's method to determine the slope of a tangent without relying on geometric construction or limits.

A different perspective was offered by Isaac Barrow, who approached tangents from a geometric point of view (Edwards, 1979; Grabiner, 1983; Strong, 1970). Strong (1970) explains that Barrow emphasized the relationship between curves and their slopes using geometric arguments. His work helped to connect the idea of tangents with broader concepts of change and laid important groundwork for later developments in calculus. Barrow's approach to tangents can be understood through the example below.

Example 3: Determining the Tangent Using Barrow's Geometric Approach

For the function $y = x^2$, the slope of the tangent at $x = 2$ is determined using the following steps.

Step 1: Take two points on the curve.

Let the first point be $x = 2 \rightarrow y = 4$, and the second point be $x = 2 + h$

Thus:

$$y = (2 + h)^2 = 4 + 4h + h^2$$

Step 2: Form the slope of the secant line

$$\frac{(4 + 4h + h^2) - 4}{h}$$

Step 3: Simplify

$$\frac{4h + h^2}{h} = 4 + h$$

Step 4: Geometric reasoning

As the second point moves closer to the first point, the secant line approaches the tangent line. When h becomes very small, the slope approaches:

$$4 + h \approx 4$$

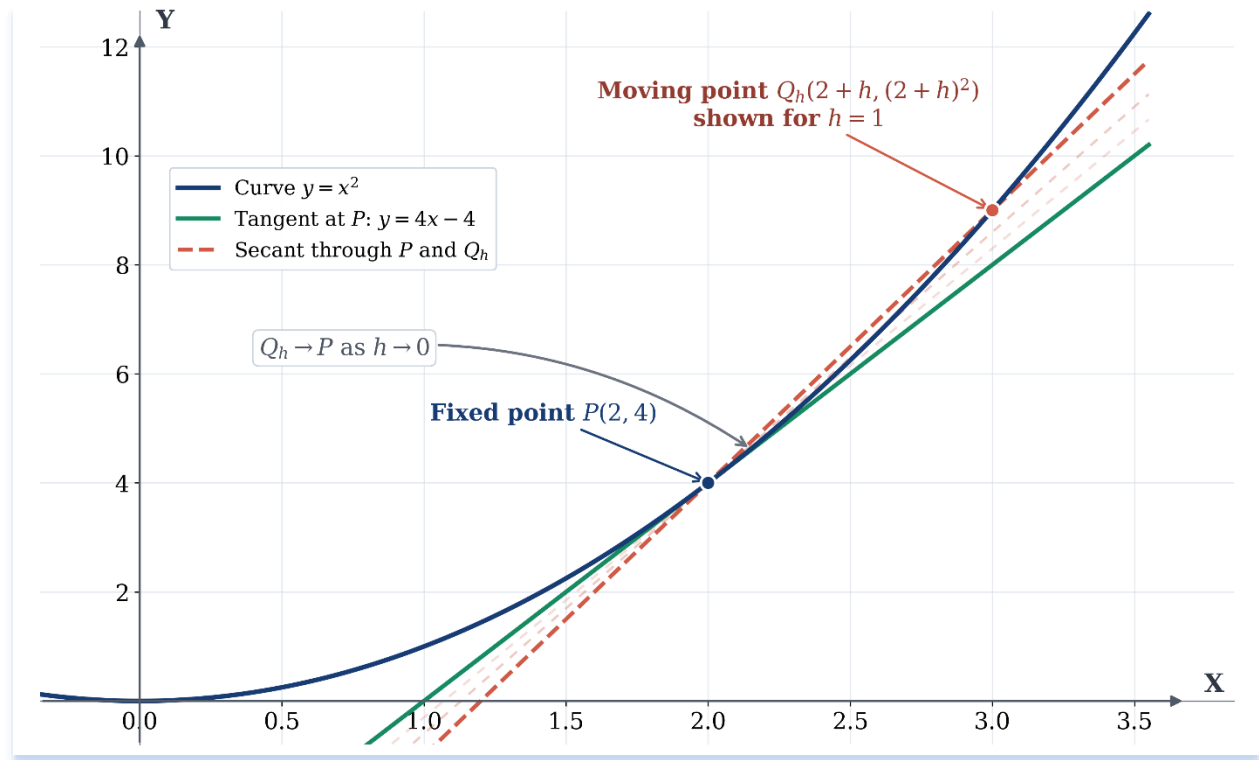
$$m = 4$$

Figure 1 illustrates Barrow's geometric interpretation by showing how the secant line through P and Q_h approaches the tangent line as Q_h moves toward P and h approaches zero.

The work of Gottfried Wilhelm Leibniz introduced a new way of thinking about tangents through infinitesimals (Bos, 1974; Brown, 2012; Scriba, 1964). Leibniz expressed the slope of a tangent line using the notation $\frac{dy}{dx}$, which represented the ratio of two infinitely small changes. Bos (1974) explains that this approach allowed for a more general and flexible treatment of tangents. Roero (2005) further notes that Leibniz's notation and ideas played a key role in shaping the development of differential calculus. Leibniz's introduction of the notation $\frac{dy}{dx}$ provided a symbolic way of expressing the slope of a tangent line, which closely aligns with the modern derivative.

Figure 1

Barrow's geometric representation of a secant line approaching the tangent line as $h \rightarrow 0$ (authors' own elaboration).



This example reflects Barrow's use of geometric reasoning to describe the behavior of tangents.

These classical approaches show that tangents were understood in different ways depending on the method used. Some approaches focused on approximation, others on algebraic procedures, and others on geometric relationships. Studies by Suzuki (2005) and Kleiner (2001) highlight that these methods were able to solve tangent problems effectively without the formal use of limits. This suggests that classical approaches offer valuable conceptual insights into the nature of tangents and the idea of change in mathematics.

Contemporary Understanding of Tangents

In contemporary mathematics, the tangent line is defined using the derivative (Bressoud, 2022; Spivak, 2018; Trigueros et al., 2025; Zill & Wright, 2009). The derivative gives the slope of a curve at a specific point and is based on the concept of a limit (Grossman, 2014; Spivak, 2018; Taylor, 1942). Standard calculus texts explain that the tangent line is obtained by finding the limit of the slopes of secant lines as the points come closer together (Grossman, 2014; Zill & Wright, 2009). This approach provides a clear and general method that can be applied to many types of functions.

Research by Grabiner (1983) explains that this modern definition of the derivative represents a shift toward formal and rigorous reasoning in mathematics. The use of limits allows mathematicians to define tangents precisely, avoiding the ambiguity found in earlier methods based on approximation or infinitesimals (Boyer, 2012; Grabiner, 1983; Kleiner, 2001). Similarly, Boyer (2012) notes that the development of limit-based calculus helped unify different approaches to tangents into a single, systematic framework.

In modern classrooms, the tangent line is often introduced through symbolic rules and procedures (Tall, 1985; Tall, 2012; Zill & Wright, 2009). For example, students are taught to differentiate functions, such as finding $\frac{dy}{dx} = 3x^2$ for $y = x^3$, and then substituting a value to obtain the slope of the tangent at a point (Grossman, 2014; Spivak, 2018; Zill & Wright, 2009). While this approach is efficient, it may not always support deep conceptual understanding. Many students can apply differentiation rules correctly but struggle to explain what a tangent line represents, often developing incomplete or incorrect concept images of tangents (Hogue & Scarcelli, 2022; Vincent et al., 2015). This suggests that procedural fluency alone does not ensure that learners understand tangency as a local relationship between a curve, its changing slope, and a line at a selected point.

Contemporary research in mathematics education has also explored graphical and numerical approaches to understanding tangents (Illanes et al., 2022; Ozaltun-Celik, 2021; Roorda, 2010; Tall, 2012). For example, Ozaltun-Celik (2021) shows that using graphs and dynamic representations can help students see how the slope of a curve changes at a point. Numerical representations can support this understanding by showing how secant slopes approach a particular value, while graphical representations make the movement of the secant line toward the tangent line visible. Symbolic representations then express this relationship through the limit and derivative notation. Learners therefore develop meaning by connecting these representations rather than treating them as separate procedures (Illanes et al., 2022; Ozaltun-Celik, 2021; Tall, 2012). These approaches aim to connect the formal definition of the derivative with visual and intuitive understanding (Tall, 2012; Vincent et al., 2015). However, even with these methods, students may still rely heavily on procedures rather than fully understanding tangency.

The contemporary approach to tangents is characterized by formal definitions, symbolic methods, and general applicability. While this approach provides precision and consistency, studies suggest that it can lead to a focus on procedures rather than meaning. An important difference emerges between contemporary and classical approaches, where earlier methods emphasized intuitive and conceptual reasoning about curves and their slopes. Unlike the classical approaches, which began with different geometric, algebraic, or infinitesimal interpretations, contemporary instruction usually brings these representations together under the formal definition of the derivative. This procedural approach is illustrated in the example below.

Example 4: Finding the Equation of a Tangent Line Using Derivatives

Given the function: $y = x^2$, to find the slope of the tangent at $x = 2$ using the derivative method, the following steps are applied.

Step 1: Differentiate

$$\frac{dy}{dx} = 2x$$

Step 2: Substitute

$$\frac{dy}{dx} = 2(2) = 4$$

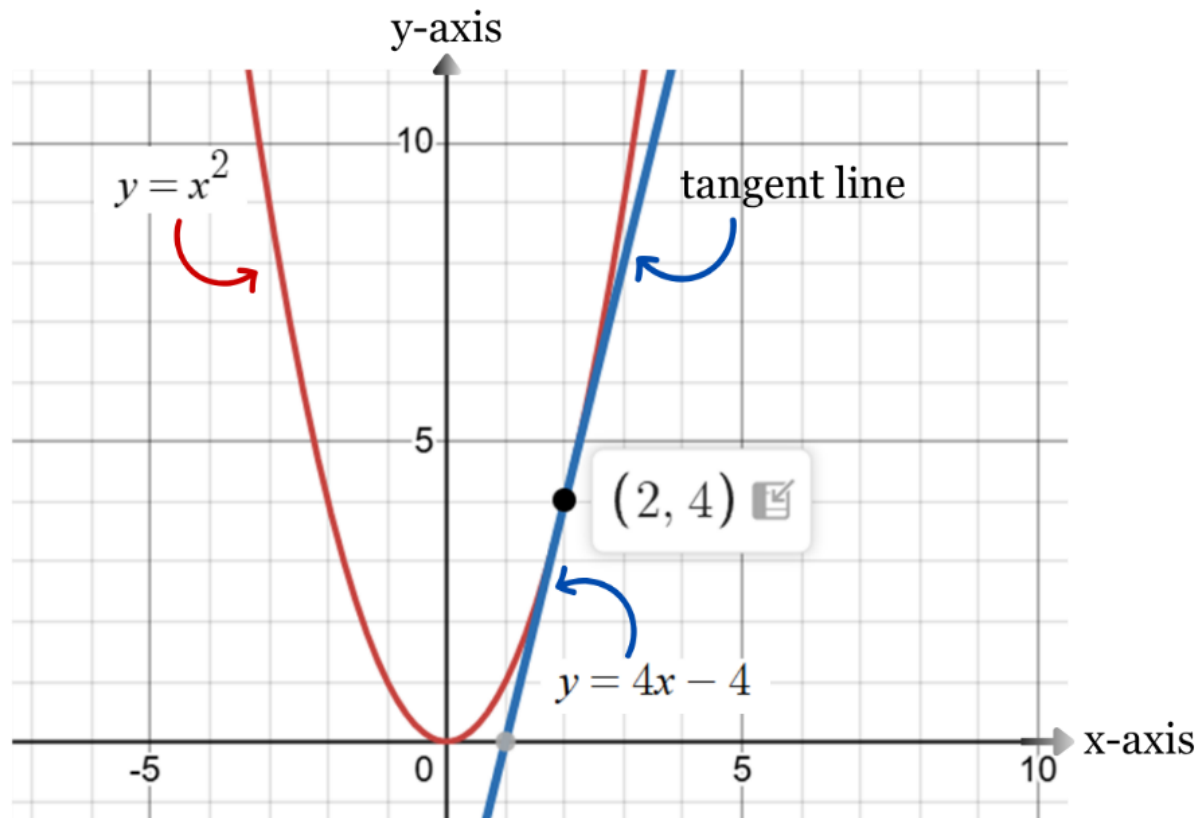
Final Answer:

$$m = 4$$

This result reflects the use of formal differentiation to determine the slope of a tangent at a point. The graphical representation in Figure 2 shows that the tangent line has the same instantaneous slope as the curve at the selected point, connecting the symbolic derivative with the local behavior of the curve.

Figure 2

Tangent line to $y = x^2$ at $x = 2$ (authors' own elaboration)



Theoretical Framework

This study is guided by the mathematical proficiency framework, which focuses on how mathematical ideas are understood beyond procedures and formulas (Kilpatrick et al., 2001; Tall, 1985; Tall, 2012; Vincent et al., 2015).

Conceptual understanding means that a learner can explain a concept, connect it to other ideas, and apply it in different contexts (Hogue & Scarcelli, 2022; Tall, 2012; Vincent et al., 2015). Research by Tall (2012) explains that understanding in mathematics develops when learners move from intuitive thinking to more formal and symbolic reasoning. This is important for concepts like tangents, which can be approached in different ways.

In the context of this study, this framework is useful because it allows researchers to examine how different mathematical traditions understand the idea of a tangent line (Grabiner, 2010; Katz, 1986; Tall, 2012). Classical mathematicians like Pierre de Fermat, René François de Sluse, Isaac Barrow, and Gottfried Wilhelm Leibniz used methods based on intuition, geometry, and algebra (Bos, 1997; Edwards, 1979; Suzuki, 2005). These approaches show how mathematical ideas were understood before formal definitions were developed. In contrast, contemporary calculus uses limits and derivatives, which represent a more formal and symbolic way of understanding tangents (Grabiner, 1983; Boyer, 2012).

The mathematical proficiency framework helps to explain this shift by showing how conceptual understanding and procedural fluency work together (Kilpatrick et al., 2001; Tall, 1985; Tall, 2012; Zuccheri et al., 2013). It also allows this study to compare how tangents are understood in different ways, rather than only comparing procedures. Research by Hogue and Scarcelli (2022) shows that students often learn procedures without fully understanding concepts. This supports the need to examine how different approaches to tangents can support deeper understanding, allowing the study to explain differences in understanding rather than only differences in methods.

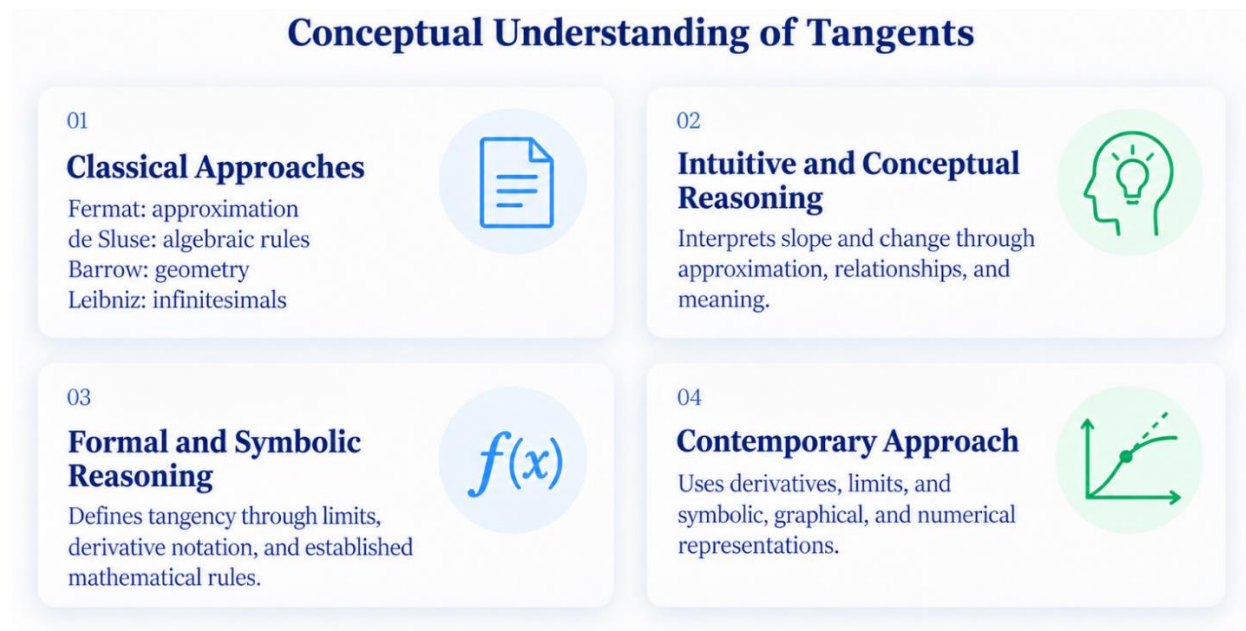
This framework is selected because it aligns with the aim of the study, which is to compare the conceptual meanings of tangents in classical and contemporary mathematics. Conceptual understanding is treated as a key component of mathematical proficiency, not as a theory on its own. It provides a basis to analyze how ideas of slope, change, and tangency are developed and represented across different approaches. Figure 3 presents a conceptual model showing how classical and contemporary approaches to tangents can be understood, highlighting the development of conceptual understanding and procedural fluency across these approaches.

Methodology

This study uses a qualitative conceptual and historical analysis to examine how the concept of a tangent line is understood in classical and contemporary mathematics. This approach is appropriate because the study does not aim to collect numerical data, but to analyze meanings, ideas, and methods across different mathematical traditions. Research in the history of mathematics supports the use of conceptual analysis to understand how mathematical ideas develop over time (Boyer, 2012; Grabiner, 1983). The data for this study consist of documented sources, including historical texts and scholarly literature on tangent methods. These include works related to Pierre de Fermat, René François de Sluse, Isaac Barrow, and Gottfried Wilhelm Leibniz, as well as modern calculus sources and mathematics education studies. These sources were selected because they provide clear explanations of tangent methods and their underlying concepts.

Figure 3

Conceptual Understanding of Tangents Across Mathematical Approaches (authors' own elaboration)



Data collection involved reviewing and selecting relevant literature that explains how tangents are defined, interpreted, and applied. The selection focused on sources that describe classical methods of tangents and those that explain the modern derivative-based approach. This ensured that both historical and contemporary perspectives were represented. The data were analyzed using a comparative conceptual analysis. This involved identifying key ideas related to tangents, including slope, change, approximation, and formal definition. These ideas were then compared across classical and contemporary approaches to identify similarities and differences. This method allows the study to focus on meaning rather than procedures.

The coding categories were developed from the mathematical proficiency framework and the forms of reasoning identified in the reviewed literature. The initial categories were conceptual and procedural reasoning. Further categories were added during the analysis to capture how each source represented and explained tangency. The final categories were algebraic, intuitive, symbolic, procedural, visual, geometric, conceptual, and formal reasoning. The unit of analysis was a passage, mathematical method, or worked example that explained how a tangent or its slope was obtained or interpreted.

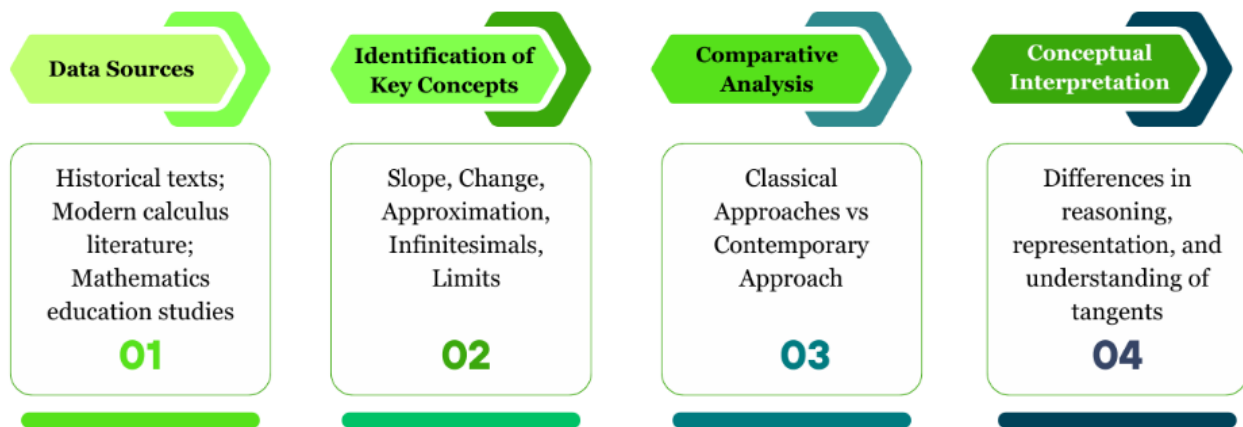
Algebraic reasoning referred to the manipulation of equations or quantities. Symbolic reasoning involved the use of mathematical notation or symbols to express tangent relationships. Procedural reasoning focused on ordered rules or computational steps used to obtain an answer. Visual reasoning used a graph, diagram, or spatial image to support interpretation, while geometric reasoning focused on relationships among curves, points, secant lines, and tangent lines. Intuitive reasoning involved informal ideas such as approximation or near-equality without a formal limit

definition. Conceptual reasoning explained the meaning of tangency, slope, or change and connected these ideas. Formal reasoning used an explicit derivative or limit definition supported by established mathematical rules.

Each unit was assigned a code when its main features matched the definition of that category. More than one code could be assigned when a method contained related forms of reasoning. For example, Fermat's method was coded as algebraic and intuitive because it combined algebraic manipulation with near-equality. De Sluse's rule was coded as symbolic and procedural because it used an algebraic rule through ordered steps. Barrow's method was coded as visual and geometric because it described the movement and relationship of secant and tangent lines. Leibniz's method was coded as symbolic and conceptual because $\frac{dy}{dx}$ represented a relationship between small changes. The contemporary derivative-based approach was coded as formal and procedural because it used the limit definition together with established differentiation rules. These coding decisions were used to organize the comparisons reported in Tables 1 and 2. To guide the analysis, a conceptual comparison framework was developed, as shown in Figure 4.

Figure 4

Conceptual Comparison Framework for Tangent Approaches (authors' own elaboration)



Results

This section presents the findings of the study based on the comparative conceptual analysis of tangent methods. Organized around the research questions, the results outline how tangents were conceptualized in classical mathematical approaches and how they are understood in contemporary calculus using derivatives and limits. Emphasis is placed on key ideas identified in the literature, including slope, change, approximation, and formal definition.

Classical Conceptualization of Tangents

The work of Fermat shows that tangents were determined using approximation (Jensen, 1969; Strømholm, 1968). This method treated the tangent as a result of comparing two nearly equal quantities. As demonstrated in Example 1, Fermat

introduced a small change to the variable, compared the new expression with the original expression, and removed the small quantity after simplification. The slope of 4 is therefore obtained by examining how the function changes near the selected point.

The analysis further shows that algebraic rules were developed for finding tangents to curves (Coolidge, 1951; Edwards, 1979; Kusraev & Kutateladze, 2017). These rules allowed tangents to be calculated directly from equations without the need for geometric construction (Coolidge, 1951). In Example 2, de Sluse's rule uses the variables and their exponents to calculate the slope directly from the equation of the curve. This approach represents a shift toward symbolic manipulation in determining slopes.

Another approach, as seen in the work of Barrow, emphasized geometric relationships between curves and their slopes (Edwards, 1979; Grabiner, 1983; Strong, 1970). His approach focused on the connection between curves and their slopes, using geometric reasoning to describe how tangents behave (Strong, 1970). Example 3 illustrates this relationship by using two points on a curve. As the second point moves closer to the first, the secant line changes position and approaches the tangent line. The changing secant slopes provide a visual explanation of how the tangent slope is formed.

Further evidence shows that Gottfried Wilhelm Leibniz introduced infinitesimals to represent change (Bos, 1974; Brown, 2012; Scriba, 1964). The slope of a tangent line was expressed as the ratio $\frac{dy}{dx}$, representing very small changes (Bos, 1974; Roero, 2005). This provided a general way of describing tangents using symbolic notation. The quantities dx and dy represent small changes in the variables, while their ratio represents the corresponding change in the curve. These results show that classical approaches to tangents were based on multiple ways of understanding slope and change (Bos, 1997; Edwards, 1979; Suzuki, 2005). These approaches did not rely on limits, but instead used approximation, algebraic procedures, geometric reasoning, and infinitesimals (Baron, 1969; Edwards, 1994; Kleiner, 2001). Fermat represented change through nearly equal quantities, de Sluse through algebraic calculation, Barrow through changing secant lines, and Leibniz through the ratio of infinitesimal changes. Thus, the interpretation of tangents as expressions of change is grounded in the specific procedures and representations identified in the analyzed examples. Table 1 presents a summary of how classical mathematicians conceptualized tangents using different methods and forms of reasoning.

Contemporary Conceptualization of Tangents

The analysis identified the limit-based definition as a central feature of the contemporary approach. The tangent line is defined as the slope of a curve at a specific point, obtained by taking the limit of the slopes of secant lines (Grossman, 2014; Spivak, 2018; Taylor, 1942). This provides a general and precise method for determining tangents for a wide range of functions (Grossman, 2014; Zill & Wright, 2009).

Table 1*Summary of Classical Approaches to Tangents*

Mathematician	Method Used	Key Concept	Nature of Reasoning	Illustrative Evidence from the Analysis
Fermat	Adequality	Approximation of values	Algebraic and intuitive	Compared nearly equal expressions after introducing a small increment.
de Sluse	Algebraic rule	Direct computation of slope	Symbolic and procedural	Applied an ordered algebraic rule directly to the curve's equation
Barrow	Geometric approach	Relationship between curve and slope	Visual and geometric	Used a moving secant line to approach the tangent line.
Leibniz	Infinitesimals ($\frac{dy}{dx}$)	Ratio of small changes	Symbolic and conceptual	Expressed slope as a ratio of infinitesimal changes.

A second pattern identified in the sources was the use of formal reasoning. The modern definition of a tangent is grounded in formal reasoning (Boyer, 2012; Grabiner, 1983; Kleiner, 2001). The introduction of limits allowed mathematicians to define change and slope rigorously (Grabiner, 1983). This development also unified earlier methods into a single framework based on differentiation (Boyer, 2012).

The reviewed sources also highlighted the use of symbolic procedures (Tall, 1985; Tall, 2012; Zill & Wright, 2009). Students are typically taught rules for differentiation and how to apply formulas to find the slope of a tangent line (Grossman, 2014; Spivak, 2018; Zill & Wright, 2009). These procedures allow for efficient computation and can be applied to many different types of functions (Bressoud, 2022; Spivak, 2018; Zill & Wright, 2009).

Graphical and numerical interpretations emerged as additional representations of tangency (Ozaltun-Celik, 2021; Roorda, 2010; Tall, 2012). Graphical representations illustrate how the slope of a curve changes at a point (Ozaltun-Celik, 2021). However, students often rely on procedures and may struggle to explain the meaning of a tangent line beyond calculation (Hogue & Scarcelli, 2022; Vincent et al., 2015).

The findings indicate that contemporary approaches to tangents are based on the derivative and the use of limits (Bressoud, 2022; Spivak, 2018; Zill & Wright, 2009). The analyzed sources conceptualize the tangent as the result of a limit process expressed through derivatives and symbolic rules. They also include graphical and numerical representations that illustrate the movement of secant lines and the corresponding changes in slope. This approach provides precision, generality, and consistency but is mainly characterized by formal definitions and procedural methods. Table 2 presents a summary of how tangents are conceptualized in contemporary calculus based on the derivative and limit definition.

Table 2*Summary of Contemporary Approach to Tangents*

Aspect	Description	Illustrative Evidence
Definition	Tangent as the slope of a curve at a point using the derivative	The derivative evaluated at the selected point gives the tangent slope.
Key Concept	Limit of secant lines approaching a point	Secant slopes approach the tangent slope as the two points move closer.
Method	Differentiation rules and formulas	The function is differentiated before the selected value is substituted.
Representation	Symbolic, graphical, and numerical	Equations, graphs, and secant-slope values represent tangency.
Nature of Reasoning	Formal and procedural	The limit definition is combined with ordered differentiation steps.

Discussion

Discussion of Classical Conceptualizations

The findings suggest that classical approaches to tangents are strongly grounded in intuitive and conceptual reasoning (Bos, 1997; Edwards, 1979; Suzuki, 2005). The methods developed by classical mathematicians show that tangents were not originally understood through formal definitions, but through meaningful interpretations of change (Baron, 1969; Edwards, 1994; Kleiner, 2001).

For example, Fermat's method of adequality can be seen as an early attempt to capture the idea of a changing quantity through approximation. As shown in Example 1, the slope emerges by comparing the original function value with the value produced after a small change in the variable. This comparison makes the relationship between change in the input and change in the output visible. This aligns with the view of Grabiner (1983), who argues that early calculus developed from attempts to understand variation before formal limits were introduced. Similarly, Leibniz's use of infinitesimals reflects a way of thinking about change as a continuous process, which supports conceptual understanding even though it lacks formal rigor (Bos, 1974). The ratio $\frac{dy}{dx}$ expresses the relationship between two small changes and therefore presents tangent slope as a rate of change rather than only a calculated value.

The work of de Sluse and Barrow also highlights the importance of different forms of reasoning. De Sluse's algebraic rules suggest a move toward symbolic thinking, while Barrow's geometric approach emphasizes visual understanding of tangents (Coolidge, 1951; Edwards, 1979; Strong, 1970). In Example 2, de Sluse's method connects the equation of the curve directly to its slope through an algebraic rule. In Example 3, Barrow's approach shows a secant line changing position as one point moves closer to another. The resulting change in the secant slope provides a visual path toward the tangent slope. These differences show that classical methods do not follow a single way of thinking but instead offer multiple perspectives on the same concept. Research by Boyer (2012) supports this view by showing that early developments in calculus were diverse and problem driven.

From the perspective of the mathematical proficiency framework (Kilpatrick et al., 2001), these classical approaches can be interpreted as supporting the development of conceptual understanding before formalization. The examples present tangents as representations of change in different ways: Fermat uses nearly equal quantities, de Sluse connects an equation to its slope, Barrow uses changing secant lines, and Leibniz uses the ratio of infinitesimal changes. Taken together, these representations may help learners interpret tangent slope as a relationship between changing quantities rather than only as the result of a procedure (Tall, 1985; Tall, 2012; Vincent et al., 2015). Because the present study did not examine learners, this interpretation describes the conceptual opportunities offered by the methods rather than measured effects on learners' understanding. This suggests that classical methods provide a strong conceptual foundation, even if they do not meet the standards of modern rigor. The discussion shows that classical approaches contribute to the conceptual interpretation of tangents by emphasizing intuition, multiple representations, and flexible reasoning. These features highlight the value of revisiting historical methods to support conceptual learning in mathematics. This reflects the role of conceptual understanding as a key component of mathematical proficiency in classical approaches.

Discussion of Contemporary Conceptualizations

Based on the results, contemporary approaches to tangents are grounded in formal and symbolic reasoning (Bressoud, 2022; Spivak, 2018; Zill & Wright, 2009). The use of limits and derivatives provides a precise and general way of defining the slope of a curve at a point. This supports the development of mathematical rigor, which is a key feature of modern calculus. Research by Grabiner (1983) explains that the introduction of limits marked an important shift toward formal definitions in mathematics. Similarly, Boyer (2012) notes that this approach unified earlier methods into a consistent theoretical framework.

A limitation of this formal approach is that it can restrict conceptual understanding when it is presented mainly through procedures. Studies by Hogue and Scarcelli (2022) indicate that students often learn how to differentiate functions but struggle to explain what a tangent line represents. Vincent et al. (2015) report similar findings, showing that students may develop incomplete concept images of tangents when instruction focuses on symbolic manipulation.

Viewed through the mathematical proficiency framework (Kilpatrick et al., 2001), contemporary approaches emphasize procedural fluency and formal symbolic reasoning (Tall, 1985; Tall, 2012; Zuccheri et al., 2013). While this is important for advanced mathematical thinking, it may not fully support the initial development of meaning. This suggests that the modern approach, although precise, may overlook the intuitive and conceptual foundations that were present in classical methods.

The discussion also indicates that attempts to include graphical and numerical representations can improve understanding, but they do not always address the deeper conceptual issues. Research by Ozaltun-Celik (2021) shows that visual approaches can help students see how the slope of a curve changes, but students may still rely on procedures

rather than developing a full understanding of tangency. The discussion suggests that contemporary approaches provide clarity, precision, and general applicability, but may lead to a procedural view of tangents if not supported by conceptual explanations. This highlights the importance of integrating different approaches to support a more complete understanding of tangents, which requires both conceptual understanding and procedural fluency.

Limitations

This study is based on selected historical and contemporary sources, which means that not all possible works on tangents were included. The interpretation of classical methods may also be influenced by how later authors explain the original ideas, rather than using only original texts. In addition, the study does not include empirical data from students or classrooms, so the findings are limited to conceptual and theoretical analysis. Future research could use an experimental design to compare how classical, contemporary, and combined approaches support learners' conceptual understanding of tangents and slopes. In-depth interviews with learners and teachers could also provide insight into how they interpret and experience these approaches. Another limitation is that the study focuses on a selected group of mathematicians, namely Pierre de Fermat, René François de Sluse, Isaac Barrow, and Gottfried Wilhelm Leibniz. While these figures are important, other contributors to the development of tangents were not included, which may limit the scope of the analysis. These limitations were addressed by using well-established and authoritative sources in the history of mathematics, including work by Boyer (2012) and Grabiner (1983). Multiple sources were compared to ensure consistency and accuracy of interpretation. The study also focused on key mathematicians who are widely recognized in the development of tangent methods, which helps maintain the relevance and strength of the analysis.

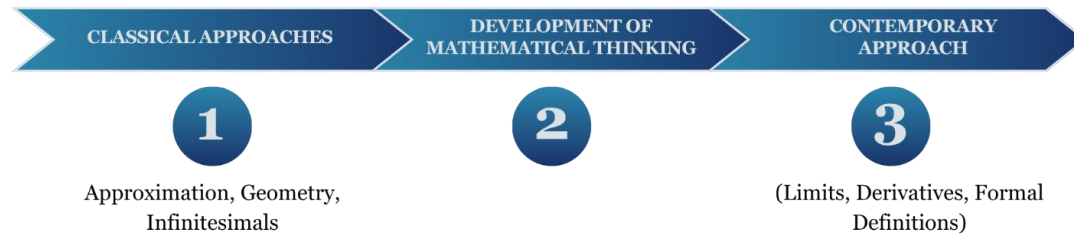
Conclusion

This study examined how the concept of a tangent line has been understood across classical and contemporary mathematics. The analysis shows that the idea of a tangent is not fixed, but has developed through different ways of thinking about change. Classical approaches describe tangents using approximation, geometric relationships, and infinitesimals, while contemporary calculus defines tangents using limits and derivatives. The study shows that these approaches are not simply different methods, but represent different forms of mathematical thinking. Classical methods focus on meaning, interpretation, and the behavior of curves, while contemporary methods focus on formal definitions and general procedures. This difference suggests that the modern definition of a tangent line does not replace earlier ideas, but builds on them in a more formal way. This supports the view of Grabiner (1983), who argues that calculus developed from earlier attempts to understand change. A key contribution of this study is that it highlights the importance of viewing tangents as a concept with multiple meanings, rather than a single definition. By bringing together classical and contemporary perspectives, the study shows that understanding tangents requires both intuitive and formal reasoning. This extends the work of David Tall (2012), which emphasizes the need to connect different forms of mathematical thinking. The findings suggest that a complete understanding of tangents should include both historical and modern perspectives. Classical approaches provide insight into how the idea of change can be understood, while contemporary methods provide precision and generality. Considering both approaches together

offers a more complete view of the concept of a tangent line. Figure 5 presents a summary of the conceptual development of tangents across mathematical approaches.

Figure 5

Conceptual Evolution of the Tangent Line (authors' own elaboration)



Implications and Recommendations

The findings of this study have important implications for both teaching practice and mathematics education. The results suggest that teaching tangents only through differentiation rules and procedures may limit students' understanding of the concept. Tall (2012) explains that meaningful learning in mathematics requires both intuitive and formal reasoning. This means that students need to understand what a tangent represents, not just how to calculate it. Including conceptual explanations of change, slope, and the behavior of curves can support deeper understanding.

The study also shows that classical approaches offer valuable insights that can support learning. The methods developed by Pierre de Fermat, René François de Sluse, Isaac Barrow, and Gottfried Wilhelm Leibniz present different ways of understanding tangents through approximation, geometry, and infinitesimals. These approaches can help students see that mathematical ideas develop from meaningful reasoning and not only from formal definitions. This aligns with the view of Grabiner (1983), who shows that calculus concepts emerged from earlier ways of thinking about change.

The findings further suggest that teaching should include multiple representations of tangents. Studies by Hogue and Scarcelli (2022) and Vincent et al. (2015) show that students often struggle to explain the meaning of a tangent line when learning is focused only on symbolic procedures. Including graphical and intuitive approaches can help students connect the formal definition of the derivative with the idea of a changing slope.

Based on these findings, it is recommended that teaching approaches combine conceptual understanding with procedural skills. Historical methods can be used to introduce the idea of tangents in a way that supports meaning before formal definitions are introduced. Instruction should also encourage reasoning, explanation, and interpretation, rather than focusing only on correct answers. Further research could use an experimental design to compare how classical, contemporary, and combined approaches support learners' conceptual understanding of tangents and slopes.

Such research could include in-depth interviews with learners and teachers to examine how they understand and experience each approach.

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